ELEMENTARY ALGEBRA

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Elementary Algebra

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TABLE OF CONTENTS

Licensing

1: The Arithmetic of Numbers
- 1.1: An Introduction to the Integers
- 1.2: Order of Operations
- 1.3: The Rational Numbers
- 1.4: Decimal Notation
- 1.5: Algebraic Expressions
- 1.E: The Arithmetic of Numbers (Exercises)

2: Solving Linear Equations and Inequalities
- 2.1: Solving Equations - One Step
- 2.2: Solving Equations - Multiple Steps
- 2.3: Clearing Fractions and Decimals
- 2.4: Formulae
- 2.5: Applications
- 2.6: Inequalities
- 2.E: Solving Linear Equations and Inequalities (Exercises)

3: Introduction to Graphing
- 3.1: Graphing Equations by Hand
- 3.2: The Graphing Calculator
- 3.3: Rates and Slope
- 3.4: Slope-Intercept Form of a Line
- 3.5: Point-Slope Form of a Line
- 3.6: Standard Form of a Line
- 3.E: Introduction to Graphing (Exercises)

4: Systems of Linear Equations
- 4.1: Solving Systems by Graphing
- 4.2: Solving Systems by Substitution
- 4.3: Solving Systems by Elimination
- 4.4: Applications of Linear Systems
- 4.E: Systems of Linear Equations (Exercises)

5: Polynomial Functions
- 5.1: Functions
- 5.2: Polynomials
- 5.3: Applications of Polynomials
- 5.4: Adding and Subtracting Polynomials
- 5.5: Laws of Exponents
- 5.6: Multiplying Polynomials
- 5.7: Special Products
- 5.E: Polynomial Functions (Exercises)

6: Factoring

- 6.1: The Greatest Common Factor
- 6.2: Solving Nonlinear Equations
- 6.3: Factoring $ax^2 + bx + c$ when $a = 1$
- 6.4: Factoring $ax^2 + bx + c$ when $a \neq 1$
- 6.5: Factoring Special Forms
- 6.6: Factoring Strategy
- 6.7: Applications of Factoring
- 6.E: Factoring (Exercises)

7: Rational Expressions

- 7.1: Negative Exponents
- 7.2: Scientific Notation
- 7.3: Simplifying Rational Expressions
- 7.4: Solving Rational Equations
- 7.5: Direct and Inverse Variation
- 7.E: Rational Expressions(Exercises)

8: Quadratic Functions

- 8.1: Introduction to Radical Notation
- 8.2: Simplifying Radical Expressions
- 8.3: Completing the Square
- 8.4: The Quadratic Formula
- 8.E: Quadratic Functions (Exercises)

Index

Glossary

Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing****.*

CHAPTER OVERVIEW

1: The Arithmetic of Numbers

1.1: An Introduction to the Integers
1.2: Order of Operations
1.3: The Rational Numbers
1.4: Decimal Notation
1.5: Algebraic Expressions
1.E: The Arithmetic of Numbers (Exercises)

Thumbnail: The addition 1+2 on the real number line. (CC BY-SA 3.0; Stephan Kulla);

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1.1: An Introduction to the Integers

We begin with the set of counting numbers, formally called the set of natural numbers.

The Natural Numbers

The set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

is called the set of natural numbers.

If we add the number zero to the set of natural numbers, then we have a set of numbers that are called the whole numbers.

The Whole Numbers

The set

$$W = \{0, 1, 2, 3, 4, 5, \ldots\}$$

is called the set of whole numbers.

The number 0 is special, in that whenever you add it to another whole number, you get the identical number as an answer.

Additive Identity Property

If a is any whole number, then

$$a + 0 = a$$

For this reason, the whole number 0 is called the additive identity.

Thus, for example, $3 + 0 = 3$, $15 + 0 = 15$, and $123 + 0 = 123$. These are all examples of the additive identity property. Every natural number has an opposite, so that when you add them together, their sum is zero.

Additive Inverse Property

If a is any natural number, then define the *opposite* of a, symbolized by $-a$, so that

$$a + (-a) = 0$$

The number $-a$ is called the "opposite of a," or more formally, the *additive inverse* of a.

For example, the opposite (additive inverse) of 3 is -3, and $3 + (-3) = 0$. The opposite (additive inverse) of 12 is -12, and $12 + (-12) = 0$. The opposite of 254 is -254, and $254 + (-254) = 0$. These are all examples of additive inverses and the additive inverse property.

Because $7 + (-7) = 0$, we've said that -7 is the opposite (additive inverse) of 7. However, we can also turn that around and say that 7 is the opposite of -7. If we translate the phrase "the opposite of -7 is 7" into mathematical symbols, we get $-(-7) = 7$.

The opposite of the opposite

Because $a + (-a) = 0$, we can say that a is the opposite of $-a$. In symbols, we write:

$$-(-a) = a$$

Thus, for example, $-(-11) = 11$, $-(-103) = 103$, and $-(-1255) = 1255$.

The Integers

If we collect all the natural numbers and their additive inverses, then include the number zero, we have a collection of numbers called the integers.

The Integers

The set

$$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

is called the set of integers.

The integers can be made to correspond to points on a line in a very natural manner. First, draw a line, then locate the number zero anywhere you wish. Secondly, place the number one to the right of zero. This determines the length of one unit. Finally, locate the numbers $1, 2, 3, 4, 5, \ldots$ to the right of zero, then their opposites (additive inverses) $-1, -2, -3, -4, -5, \ldots$ to the left of zero (see Figure $1.1.1$).

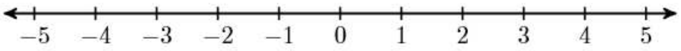


***Figure** $1.1.1$**:** Each integer corresponds to a unique position on the number line.*

Note that as we move to the right on the number line, the integers get larger. On the other hand, as we move to the left on the number line, the integers get smaller.

Positive and negative integers

On the number line, some integers lie to the right of zero and some lie to the left of zero.

- If a is an integer that lies to the right of zero, then a is called a *positive* integer.
- If a is an integer that lies to the left of zero, then a is called a *negative* integer.

Thus, 4, 25, and 142 are positive integers, while -7, -53, and -435 are negative integers.

Absolute Value

The absolute value (or magnitude) of an integer is defined as follows.

The Absolute Value of an Integer

If a is an integer, then the absolute value of a, written $|a|$, is defined as the distance between the integer and zero on the number line.

Example $1.1.1$

Simplify $|-4|$.

Solution

Consider the position of -4 on the number line. Note that -4 lies four units away from zero.

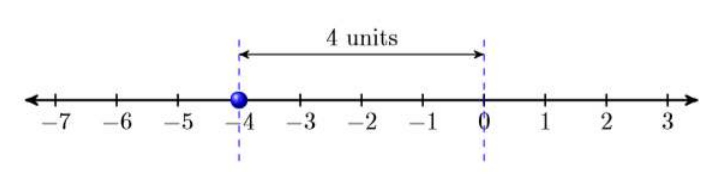


Figure $1.1.2$

Because the absolute value (magnitude) of an integer equals its distance from zero, $|-4| = 4$.

Exercise $1.1.1$

Simplify: $|-23|$

Answer

$|-23| = 23$

In similar fashion:

- The integer 5 lies five units away from zero. Hence, $|5| = 5$.
- The integer 0 lies zero units away from zero, Hence, $|0| = 0$. Note that the absolute value of any number is either positive or zero. That is, the absolute value of a number is nonnegative (not negative).

Integer Addition

This section is designed to provide a quick review of integer addition. We consider the first of two cases.

Adding Integers with Like Signs

To add two integers with like signs (both positive or both negative), add their magnitudes (absolute values), then prefix their common sign.

Example 1.1.2

Simplify $7+12$.

Solution

We have like signs. The magnitudes (absolute values) of 7 and 12 are 7 and 12, respectively. If we add the magnitudes, we get 19. If we prefix the common sign, we get 19. That is:

$$7+12=19$$

Exercise 1.1.2

Simplify: $13+28$.

Answer

41

Example 1.1.3

Simplify $-8+(-9)$.

Solution

We have like signs. The magnitudes (absolute values) of -8 and -9 are 8 and 9, respectively. If we add the magnitudes, we get 17. If we prefix the common sign, we get -17. That is:

$$-8+(-9)=-17$$

Exercise 1.1.3

Simplify: $-12+(-21)$.

Answer

-33

Next, we consider the case where we have unlike signs.

Adding Integers with Unlike Signs

To add two integers with unlike signs (one positive and one negative), subtract the integer with the smaller magnitude (absolute value) from the number with the larger magnitude, then prefix the sign of the integer with the larger magnitude.

Example 1.1.4

Simplify $-14+11$.

Solution

We have unlike signs. The magnitudes (absolute values) of -14 and 11 are 14 and 11, respectively. If we subtract the smaller magnitude from the larger, we get 3. The number -14 has the larger magnitude, so we prefix our answer with its negative sign. That is:

$$-14+11=-3$$

Exercise 1.1.4

Simplify: $12+(-29)$.

Answer

-17

Example 1.1.5

Simplify $40+(-25)$.

Solution

We have unlike signs. The magnitudes (absolute values) of 40 and -25 are 40 and 25, respectively. If we subtract the smaller magnitude from the larger, we get 15. The number 40 has the larger magnitude, so we prefix our answer with its positive sign. That is:

$$40+(-25)=15$$

Exercise 1.1.5

Simplify: $32+(-90)$.

Answer

-58

Mathematical Properties of Addition

The order in which we add integers does not matter. That is, $-20+34$ gives an answer identical to the sum $34+(-20)$. In both cases, the answer is 14. This fact is called the *commutative property of addition*.

The Commutative Property of Addition

If a and b are any two integers, then:

$$a+b=b+a$$

Next, when we add three integers, it does not matter which two we add first. For example, if we add the second and third of three numbers first, we get:

$$\begin{aligned} -11+(-2+5) &= -11+3 && \text{Parentheses first: } -2+5=3 \\ &= -8 && \text{Add: } -11+3=-8 \end{aligned}$$

On the other hand, if we add the first and second of three numbers first, we get:

$$\begin{aligned} (-11+(-2))+5 &= -13+5 && \text{Parentheses first: } -11+(-2)=-13 \\ &= -8 && \text{Add: } -13+5=-8 \end{aligned}$$

Thus, $-11+(-2+5)=(-11+(-2))+5$. This fact is called the *associative property of addition*.

The Associative Property of Addition

If a, b, and c are any three integers, then:

$$a+(b+c)=(a+b)+c$$

Integer Subtraction

Subtraction is the inverse, or the opposite, of addition.

Subtracting Integers

If a and b are any two integers, then:

$$a-b=a+(-b)$$

Subtracting b is identical to adding the opposite (additive inverse) of b.

Example 1.1.6

Simplify: $-13-27$.

Solution

The “opposite” (additive inverse) of 27 is -27. So, subtracting 27 is the same as adding -27.

$$\begin{aligned} -13-27 &= -13+(-27) && \text{Subtracting 27 is the same as adding } -27. \\ &= -50 && \text{Add the magnitudes, then prefix the common negative sign.} \end{aligned}$$

Exercise 1.1.6

Simplify: $-11-15$.

Answer

-26

Example 1.1.7

Simplify: $-27-(-50)$.

Solution

The “opposite” (additive inverse) of -50 is $-(-50)$, or 50. So, subtracting -50 is the same as adding 50.

$$\begin{aligned} -27-(-50) &= -27+50 && \text{Subtracting -50 is the same as adding 50.} \\ &= 23 && \text{Subtract the smaller magnitude from the larger magnitude, then prefix the sign of the larger magnitude.} \end{aligned}$$

Exercise 1.1.7

Simplify: $-18-(-54)$.

Answer

36

Integer Multiplication

This section is designed to provide a quick review of multiplication and division of integers.

Like Signs

If a and b are integers with like signs (both positive or both negative), then the product ab and the quotient a/b are positive.

$$\begin{array}{ccc} (+)(+)=+ & \text{or} & (+)/(+)=+ \\ (-)(-)=+ & \text{or} & (-)/(-)=+ \end{array}$$

Example 1.1.8

Simplify each of the following expressions:

a. $(2)(3)$
b. $(-12)(-8)$
c. $-14/(-2)$

Solution

When multiplying or dividing, like signs yield a positive result.

a. $(2)(3)=6$
b. $(-12)(-8)=96$
c. $-14/(-2)=7$

Exercise 1.1.8

Simplify: $(-18)(-5)$.

Answer

90

Unlike Signs

If a and b are integers with unlike signs (one positive and one negative), then the product ab and the quotient a/b are negative.

$$\begin{array}{lcl} (+)(-)=- & \text{or} & (+)/(-)=- \\ (-)(+)=- & \text{or} & (-)/(+)=- \end{array}$$

Example 1.1.9

Simplify each of the following expressions:

a. $(2)(-12)$
b. $(-9)(12)$
c. $24/(-8)$

Solution

When multiplying or dividing, unlike signs yield a negative result.

a. $(2)(-12)=-24$
b. $(-9)(12)=-108$
c. $24/(-8)=-3$

Exercise 1.1.9

Simplify: $(-19)(3)$.

Answer

-57

Mathematical Properties of Multiplication

The order in which we multiply integers does not matter. That is, $(-8)(5)$ gives an answer identical to $(5)(-8)$. In both cases, the answer is -40. This fact is called the commutative property of multiplication.

The Commutative Property of Multiplication

If a and b are any two integers, then:

$$a \cdot b = b + a$$

Next, when we multiply three integers, it does not matter which two we multiply first. If we multiply the second and third of three numbers first, we get:

$$\begin{aligned} (-3)[(-4)(-5)] &= (-3)(20) && \text{Brackets first: } (-4)(-5)=20 \\ &= -60 && \text{Multiply: } (-3)(20)=-60 \end{aligned}$$

On the other hand, if we multiply the first and second of three numbers first, we get:

$$\begin{aligned} [(-3)(-4)](-5) &= (12)(-5) && \text{Brackets first: } (-3)(-4)=12 \\ &= -60 && \text{Multiply: } (12)(-5)=-60 \end{aligned}$$

Thus, $(-3)[(-4)(-5)]=[(-3)(-4)](-5)$. This fact is called the *associative property of multiplication*.

The Associative Property of Multiplication

If a, b, and c are any three integers, then:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

When you multiply an integer by 1, you get the identical number back as the product. For example, $(1)(5)=5$ and $(-11)(1)=-11$. This fact is known as the *multiplicative identity property*.

The Multiplicative Identity Property

If a is any integer, then:

$$1\cdot a=a \quad \text{and} \quad a\cdot 1=a$$

For this reason, the integer 1 is called the “multiplicative identity.”

Finally, note that $(-1)(5)=-5$. Thus, multiplying 5 by -1 is identical to taking the “opposite” of 5 or negating 5.

The Multiplicative Property of -1

Multiplying by minus one is identical to negating. That is:

$$(-1)a=-a$$

Exponents

In the exponential expression a^n, the number a is called the base, while the number n is called the *exponent*. We now define what is meant by an exponent.

Exponents

Let a be an integer and let n be any whole number. If $n\neq 0$, then:

$$a^n=\underbrace{a\cdot a\cdot a\cdot\cdots\cdot a}_{n\text{ times}}$$

That is, to calculate a^n, write a as a factor n times.

Example 1.1.10

Simplify $(-2)^3$.

Solution

In the exponential expression $(-2)^3$, note that -2 is the base, while 3 is the exponent. The exponent tells us to write the base as a factor three times. Simplify the result by performing the multiplications in order, moving from left to right.

$$\begin{aligned}(-2)^3&=(-2)(-2)(-2) \quad \text{-2 as a factor, three times.}\\&=(4)(-2) \quad \text{Multiply: (-2)(-2)=4}\\&=-8 \quad \text{Multiply: (4)(-2)=-8}\\ \text{Thus } (-2)^3&=-8\end{aligned}$$

Exercise 1.1.10

Simplify: $(-2)^2$.

Answer

4

In Example 1.1.10, note that the product of three negative factors is negative. Let’s try another example.

Example 1.1.11

Simplify $(-2)^4$.

Solution

In the exponential expression $(-2)4$, note that -2 is the base, while 4 is the exponent. The exponent tells us to write the base as a factor four times. Simplify the result by performing the multiplications in order, moving from left to right.

$$\begin{aligned}(-2)^4&=(-2)(-2)(-2)(-2) \quad \text{-2 as a factor, four times.}\\&=(4)(-2)(-2) \quad \text{Multiply: (-2)(-2)=4}\\&=(-8)(-2) \quad \text{Multiply: (4)(-2)=-8}\end{aligned}$$

$$= 16 \quad \text{Multiply: (-8)(-2)=-8}$$
$$\text{Thus } (-2)^4 = 16$$

Exercise 1.1.11

Simplify: $(-2)^5$.

Answer

-32

In Example 1.1.11, note that the product of four negative factors is positive. Examples 1.1.10 and 1.1.11 reveal the following pattern.

Odd or Even Exponents

1. When a negative integer is raised to an even exponent, the result is positive.
2. When a negative integer is raised to an odd exponent, the result is negative.

Graphing Calculator: Negating versus Subtracting

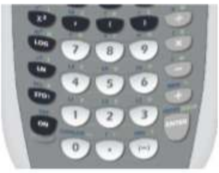


***Figure** 1.1.3**:** Lower half of the TI-84.*

Consider the view of the lower half of the TI-84 graphing calculator in Figure 1.1.3. Note that there are two keys that contain some sort of negative sign, one on the bottom row of keys, and another in the last column of keys on the right, positioned just above the plus symbol.

$\boxed{(-)}$ and $\boxed{-}$

***Figure** 1.1.4*

The first of these buttons is the unary "negation" operator. If you want to negate a single (thus the word "unary") number, then this is the key to use. For example, enter -3 by pressing the following button sequence. The result is shown in Figure 1.1.7.

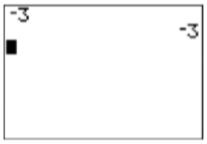
$\boxed{(-)}$ $\boxed{3}$ $\boxed{\text{ENTER}}$

***Figure** 1.1.5*

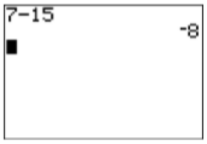
The second button is the binary "subtraction" operator. If you want to subtract one number from another number (thus the word "binary"), then this is the key to use. For example, enter $7 - 15$ by pressing the following button sequence. The result is shown in Figure 1.1.8.

$\boxed{7}$ $\boxed{-}$ $\boxed{1}$ $\boxed{5}$ $\boxed{\text{ENTER}}$

***Figure** 1.1.6*



```
-3
                -3
```



```
7-15
                -8
```

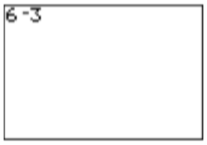
***Figure** 1.1.7**:** Negating a number. **Figure** 1.1.8**:** Subtract two numbers.*

Note

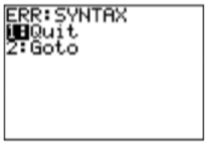
Do not interchange the roles of the unary negation operator and the binary subtraction operator.

1. To negate a number, use: (-)
2. To subtract one number from another, use: −

If you interchange the roles of these operators, the calculator will respond that you've made a "syntax error" (see Figures 1.1.9 and 1.1.10).



```
⁻6⁻3
```



```
ERR:SYNTAX
1:Quit
2:Goto
```

***Figure** 1.1.9: Using the wrong symbol for subtraction.* ***Figure** 1.1.10: The resulting syntax error.*

Example 1.1.12

Use the TI-84 graphing calculator to simplify each of the following expressons:

a. $-717-432$
b. $(232)(-313)$
c. $(-17)^3$

Solution

The minus sign in each of these examples looks exactly the same, but sometimes it is used as a "negative" sign and sometimes it is used as a "subtraction" sign.

a. The expression $-717-432$ asks us to subtract 432 from "negative" 717. Enter the following sequence of keystrokes to produce the result shown in the first image in Figure 1.1.11.

(-) 7 1 7 − 4 3 2 ENTER

Hence, $-717-432=-1149$.

b. The expression $(232)(-313)$ asks us to find the product of 232 and "negative" 313. Enter the following sequence of keystrokes to produce the result shown in the second image in Figure \(\PageIndex{11}\).

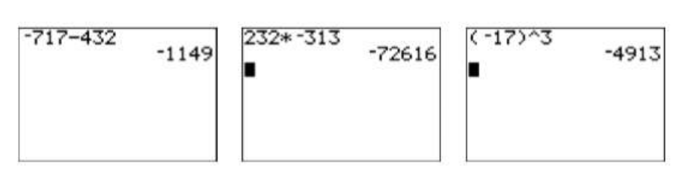
2 3 2 × (-) 3 1 3 ENTER

Hence, $(232)(-313)=-72616$.

c. The expression $(-17)^3$ asks us to raise "negative" to the third power. Enter the following sequence of keystrokes to produce the result shown in the third image in Figure 1.1.11. The "caret" symbol ^ is located just above the division key in the rightmost column of the TI-84 graphing calculator.

((-) 1 7) ^ 3 ENTER

Hence, $(-17)^3=-4913$.



```
⁻717-432
            ⁻1149
```

```
232*⁻313
           ⁻72616
```

```
(⁻17)^3
            ⁻4913
```

***Figure** 1.1.11: Calculations made on the graphing calculator.*

Exercise 1.1.12

Use the graphing calculator to evaluate $(-225)^3$.

Answer

$(-225)^3 = -11390625$

Contributors

- David Arnold (Retired Professor (Mathematics) at College of the Redwoods)

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1.2: Order of Operations

The order in which we evaluate expressions can be ambiguous. Take, for example, the expression $-4+2\cdot 8$. If we perform the addition first, then we get -16 as a result (the question mark over the equal sign indicates that the result is questionable).

$$\begin{aligned} -4+2\cdot 8 &\overset{?}{=} -2\cdot 8 \\ &\overset{?}{=} -16 \end{aligned}$$

On the other hand, if we perform the multiplication first, then we get 12 as a result.

$$\begin{aligned} -4+2\cdot 8 &\overset{?}{=} -4+16 \\ &\overset{?}{=} 12 \end{aligned}$$

So, what are we to do? Of course, grouping symbols would remove the ambiguity.

GROUPING SYMBOLS

Parentheses, brackets, and absolute value bars can be used to group parts of an expression. For example:

$$3+5(9-11) \quad \text{or} \quad -2-[-2-5(1-3)] \quad \text{or} \quad 6-3|-3-4|$$

In each case, the rule is “evaluate the expression inside the grouping symbols first.” If the grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.

Thus, if the example above is grouped as follows, we are forced to evaluate the expression inside the parentheses first.

$$\begin{aligned} (-4+2)\cdot 8 &= -2\cdot 8 && \text{Parentheses first: } -4+2=-2 \\ &= -16 && \text{Multiply: } -2\cdot 8=-16 \end{aligned}$$

Another way to avoid ambiguities in evaluating expressions is to establish an order in which operations should be performed. The following guidelines should always be strictly enforced when evaluating expressions.

RULES GUIDING ORDER OF OPERATIONS

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 1.2.1

Simplify: $-3-4\cdot 8$.

Solution

Because of the established Rules Guiding Order of Operations, this expression is no longer ambiguous. There are no grouping symbols or exponents present, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right. After that we invoke rule four, performing all additions and subtractions in the order that they appear, moving left to right.

$$-3-4\cdot 8 = -3-32 \qquad \text{Multiply first: } 4\cdot 8=32$$

$$\begin{aligned} &= -3 + (-32) \quad \text{Add the opposite.} \\ &= -35 \quad \text{Add: } -3 + (-32) = -35 \\ \text{Thus, } -3 - 4 \cdot 8 &= -35 \end{aligned}$$

Exercise 1.2.1

Simplify: $-4 + 2 \cdot 8$.

Answer

12

Writing Mathematics

When simplifying expressions, observe the following rule to neatly arrange your work:
One equal sign per line.

This means that you should not arrange your work horizontally.

$$-2 - 4 \cdot (-8) = -2 - (-32) = -2 + 32 = 30$$

That's three equal signs on a single line. Rather, arrange your work vertically, keeping equal signs aligned in a column.

$$\begin{aligned} -2 - 4 \cdot (-8) &= -2 - (-32) \\ &= -2 + 32 \\ &= 30 \end{aligned}$$

Example 1.2.2

Simplify: $54/(-9)(2)$.

Solution

There are no grouping symbols or exponents present, so we immediately go to rule three, evaluate all multiplications and divisions in the order that they appear, moving left to right.

$$\begin{aligned} 54/(-9)(2) &= -6(2) \quad \text{Divide first: } 54/(-9) = -6 \\ &= -12 \quad \text{Multiply: } -6(2) = -12 \\ \text{Thus, } 54/(-9)(2) &= -12 \end{aligned}$$

Exercise 1.2.2

Simplify: $-24/(-3)(2)$.

Answer

16

Example 1.2.2 can be a source of confusion for many readers. Note that multiplication takes no preference over division, nor does division take preference over multiplication. Multiplications and divisions have the same level of preference and must be performed in the order that they occur, moving from left to right. If not, the wrong answer will be obtained.

Warning!

Here is what happens if you perform the multiplication in Example 1.2.2 before the division.

$$\begin{aligned} 54/(-9)(2) &= 54/(-18) \quad \text{Multiply: } (-9)(2) = -18 \\ &= -3 \quad \text{Divide: } 54/(-18) = -3 \end{aligned}$$

This is incorrect! Multiplications and divisions must be performed in the order that they occur, moving from left to right.

Example 1.2.3

Simplify:

a. $(-7)^2$

b. -7^2

Solution

Recall that for any integer a, we have $(-1)a = -a$. Because negating is equivalent to multiplying by -1, the Rules Guiding Order of Operations require that we address grouping symbols and exponents before negation.

a. Because of the grouping symbols, we negate first, then square. That is,

$$\begin{aligned}(-7)^2 &= (-7)(-7)\\ &= 49\end{aligned}$$

b. There are no grouping symbols in this example. Thus, we must square first, then negate. That is,

$$\begin{aligned}-7^2 &= -(7\cdot 7)\\ &= -49\end{aligned}$$

Thus, $(-7)^2 = 49$, but $-7^2 = -49$. Note: This example demonstrates that $(-7)^2$ is different from -7^2.

Exercise 1.2.3

Simplify: -15^2.

Answer

-225

Let's try an example that has a mixture of exponents, multiplication, and subtraction.

Example 1.2.4

Simplify: $-3-2(-4)^2$.

Solution

The Rules Guiding Order of Operations require that we address exponents first, then multiplications, then subtractions.

$$\begin{aligned}-3-2(-4)^2 &= -3-2(16) \quad \text{Exponent first: } (-4)^2 = 16\\ &= -3-32 \quad \text{Multiply: } 2(16) = 32\\ &= -3+(-32) \quad \text{Add the opposite.}\\ &= -35 \quad \text{Add: } -3+(-32) = -35\\ \text{Thus, } -3-2(-4)^2 &= -35\end{aligned}$$

Exercise 1.2.4

Simplify: $-5-4(-2)^3$.

Answer

27

Grouping Symbols

The *Rules Guiding Order of Operations* require that expressions inside grouping symbols (parentheses, brackets, or curly braces) be evaluated first.

Example 1.2.5

Simplify: $-2(3-4)^2+5(1-2)^3$.

Solution

The *Rules Guiding Order of Operations* require that we first evaluate the expressions contained inside the grouping symbols.

$$\begin{aligned}-2(3-4)^2+5(1-2)^3 &= -2(3+(-4))^2+5(1+(-2))^3\\ &\qquad \text{Add the opposites.}\\ &= -2(-1)^2+5(-1)^3\\ &\qquad \text{Parenthesis first: } 3+(-4)\end{aligned}$$

$$= -1 \text{ and } 1 + (-2) = -1$$

Evaluate the exponents next, perform the multiplications, then add.

$$\begin{aligned}
&= -2(1) + 5(-1) \\
&\qquad \text{Exponents: } (-1)^2 = 1 \text{ and } (-1)^3 = -1 \\
&= -2(1) + 5(-1) \\
&\qquad \text{Multiply: } -2(1) = -2 \text{ and } 5(-1) = -5 \\
&= -7 \quad \text{Add } -2 + (-5) = -7
\end{aligned}$$

$$\text{Thus, } -2(3-4)^2 + 5(1-2)^3 = -7$$

Exercise 1.2.5

Simplify: $-2 - 3(-2-3)^3$.

Answer

373

Absolute Value Bars as Grouping Symbols

Like parentheses and brackets, you must evaluate what is inside them first, then take the absolute value of the result.

Example 1.2.6

Simplify: $-8 - |5 - 11|$.

Solution

We must first evaluate what is inside the absolute value bars.

$$\begin{aligned}
-8 - |5 - 11| &= -8 - |5 + (-11)| \quad \text{Add the opposites.} \\
&= -8 - |-6| \quad \text{Add: } 5 + (-11) = -6
\end{aligned}$$

The number -6 is 6 units from zero on the number line. Hence, $|-6| = 6$

$$\begin{aligned}
&= -8 - 6 \quad \text{Add: } |-6| = 6 \\
&= -8 + (-6) \quad \text{Add the opposite.} \\
&= -14 \quad \text{Add. } -2 + (-5) = -7
\end{aligned}$$

$$\text{Thus, } -8 - |5 - 11| = -14$$

Exercise 1.2.6

Simplify: $-|-4-6|$.

Answer

-10

Nested Grouping Symbols

When grouping symbols are nested, the *Rules Guiding Order of Operations* tell us to evaluate the innermost expressions first.

Example 1.2.7

Simplify: $-3 - 4[-3 - 4(-3-4)]$.

Solution

The Rules Guiding Order of Operations require that we first address the expression contained in the innermost grouping symbols. That is, we evaluate the expression contained inside the brackets first.

$$\begin{aligned}
-3 - 4[-3 - 4(-3-4)] &= -3 - 4[-3 - 4(-3 + (-4))] \quad \text{Add the opposite.} \\
&= -3 - 4[-3 - 4(-7)] \quad \text{Add: } -3 + (-4) = -7
\end{aligned}$$

Next, we evaluate the expression contained inside the brackets.

$$\begin{aligned}
&= -3 - 4[-3 - (-28)] \quad \text{Multiply: } 4(-7) = -28 \\
&= -3 - 4[-3 + 28] \quad \text{Add the opposite.} \\
&= -3 - 4[25] \quad \text{Add: } -3 + 28 = 25
\end{aligned}$$

Now we multiply, then subtract.

$$\begin{aligned} &= -3 - 100 && \text{Multiply: } 4\{25\} = 100 \\ &= -3 + (-100) && \text{Add the opposite.} \\ &= -103 && \text{Add: } -3 + (-100) = -103 \end{aligned}$$

Thus, $-3 - 4[-3 - 4(-3 - 4)] = -103$

Exercise 1.2.7

Simplify: $-2 - 2[-2 - 2(-2 - 2)]$.

Answer

-14

Evaluating Algebraic Expressions

Variable

A variable is a symbol (usually a letter) that stands for an unknown value that may vary.

Let's add the definition of an *algebraic expression*.

Algebraic Expression

When we combine numbers and variables in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, the resulting combination of mathematical symbols is called an *algebraic expression*.

Thus,

$$2a, \quad x + 5, \quad \text{and} \quad y^2$$

being formed by a combination of numbers, variables, and mathematical operators, are valid algebraic expressions.

An algebraic expression must be well-formed. For example,

$$2 + -5x$$

is not a *valid expression* because there is no term following the plus sign (it is not valid to write $+-$ with nothing between these operators). Similarly,

$$2 + 3(2$$

is not well-formed because parentheses are not balanced.

In this section we will evaluate algebraic expressions for given values of the variables contained in the expressions. Here are some simple tips to help you be successful.

Tips for Evaluating Algebraic Expressions

1. Replace all occurrences of variables in the expression with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the *Rules Guiding Order of Operations*.

Example 1.2.8

Evaluate the expression $x^2 - 2xy + y^2$ at $x = -3$ and $y = 2$.

Solution

Following Tips for *Evaluating Algebraic Expressions*, first replace all occurrences of variables in the expression $x^2 - 2xy + y^2$ with open parentheses. Next, substitute the given values of variables (-3 for x and 2 for y) in the open parentheses.

$$\begin{aligned} x^2 - 2xy + y^2 &= (\ \ )^2 - 2(\ \ )() + ()^2 \\ &= (-3)^2 - 2(-3)(2) + (2)^2 \end{aligned}$$

Finally, follow the *Rules Guiding Order of Operations* to evaluate the resulting expression.

$$
\begin{aligned}
&x^2-2xy+y^2 \quad \text{Original Expression.}\\
&=(\ \ )^2-2(\ \ )()+()^2\\
&\qquad \text{Replace variables with parenthesis.}\\
&=(-3)^2-2(-3)(2)+(2)^2\\
&\qquad \text{Substitute } -3 \text{ for } x \text{ and } 2 \text{ for } y\\
&=9-2(-3)(2)+4\\
&\qquad \text{Evaluate exponenets first.}\\
&=9-(-6)(2)+4\\
&\qquad \text{Left to right, multiply: } 2(-3)=-6\\
&=9-(-12)+4\\
&\qquad \text{Left to right, multiply: } (-6)2=-12\\
&=9+12+4 \quad \text{Add the opposite.}\\
&=25 \quad \text{Add.}
\end{aligned}
$$

Thus, if $x=-3$ and $y=2$, then $x^2-2xy+y^2=25$

Exercise 1.2.8

If $x=-2$ and $y=-1$, evaluate x^3-y^3.

Answer

-7

Evaluating Fractions

If a fraction bar is present, evaluate the numerator and denominator separately according to the *Rules Guiding Order of Operations*, then perform the division in the final step.

Example 1.2.9

Evaluate the expression

$$\frac{ad-bc}{a+b}$$

at $a=5, b=-3, c=2$, and $d=-4$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variables in the expression $(ad-bc)/(a+b)$ with open parentheses. Next, substitute the given values of variables (5 for a, -3 for b, 2 for c, and -4 for d) in the open parentheses.

$$
\begin{aligned}
\frac{ad-bc}{a+b} &= \frac{(x)-(x)}{()+()}\\
&= \frac{(5)(-4-)-(-3)(2)}{(5)+(-3)}
\end{aligned}
$$

Finally, follow the *Rules Guiding Order of Operations* to evaluate the resulting expression. Note that we evaluate the expressions in the numerator and denominator separately, then divide.

$$
\begin{aligned}
\frac{ad-bc}{a+b} &= \frac{(x)-(x)}{()+()} \quad \text{Replace variables with parentheses.}\\
&= \frac{(5)(-4-)-(-3)(2)}{(5)+(-3)}\\
&\qquad \text{Substitute: } 5 \text{ for } a, -3 \text{ for } b, 2 \text{ for } c, -4 \text{ for } d\\
&= \frac{-20-(-6)}{2} \quad \text{Numerator: } (5)(-4)=-20,\\
&\qquad (-3)(2)=-6, \text{ Denominator: } 5+(-3)=2\\
&= \frac{-20+6}{2} \quad \text{Numerator: Add the opposite.}\\
&= \frac{-14}{2} \quad \text{Numerator: } -20+6=-14\\
&= -7 \quad \text{Divide}
\end{aligned}
$$

Thus, if $x=-3$ and $y=2$,

then $x^2 - 2xy + y^2 = 25$

Exercise 1.2.9

If $a = -7, b = -3, c = -15$ and $d = -14$, evaluate: $\dfrac{a^2+b^2}{c+d}$

Answer

-2

Using the Graphing Calculator

The graphing calculator is a splendid tool for evaluating algebraic expressions, particularly when the numbers involved are large.

Example 1.2.10

Use the graphing calculator to simplify the following expression:

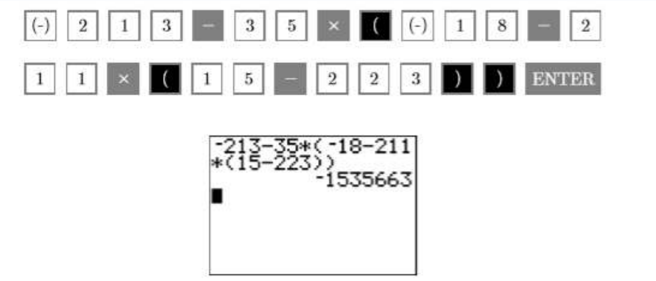
$$-213 - 35[-18 - 211(15 - 223)]$$

Solution

The first difficulty with this expression is the fact that the graphing calculator does not have a bracket symbol for the purposes of grouping. The calculator has only parentheses for grouping. So we first convert our expression to the following:

$$-213 - 35(-18 - 211(15 - 223))$$

Note that brackets and parentheses are completely interchangeable. The next difficulty is determining which of the minus signs are negation symbols and which are subtraction symbols. If the minus sign does not appear between two numbers, it is a negation symbol. If the minus sign does appear between two numbers, it is a subtraction symbol. Hence, we enter the following keystrokes on our calculator. The result is shown in Figure 1.2.1.



```
-213-35*(-18-211
*(15-223))
          -1535663
```

Figure 1.2.1: Calculating $-213 - 35[-18 - 211(15 - 223)]$

Thus, $-213 - 35[-18 - 211(15 - 223)] = -1,535,663$.

Exercise 1.2.10

Use the graphing calculator to evaluate:

$$-2 - 2[-2 - 2(-2 - 2)]$$

Answer

-14

Example 1.2.11

Use the graphing calculator to evaluate:

$$\frac{5+5}{5+5}$$

Solution

You might ask “Why do we need a calculator to evaluate this exceedingly simple expression?” After all, it’s very easy to compute.

$$\begin{aligned}\frac{5+5}{5+5}&=\frac{10}{10} && \text{Simplify numerator and denominator.}\\ &=1 && \text{Divide: } 10/10=1\end{aligned}$$

Well, let’s enter the expression $5\ +5/5\ +5$ in the calculator and see how well we understand the Rules Guiding Order of Operations (see first image in Figure $1.2.2$). Whoa! How did the calculator get 11? The answer is supposed to be 1! Let’s slow down and apply the *Rules Guiding Order of Operations* to the expression $5+5/5+5$.

$$\begin{aligned}\frac{5+5}{5+5}&=5+\frac{5}{5}+5 && \text{Divide first.}\\ &=5+1+5 && \text{Divide: } \frac{5}{5}=1\\ &=11 && \text{Add: } 5+1+5=11\end{aligned}$$

Aha! That’s how the calculator got 11.

$$5+5/5+5 \quad \text{is equivalent to} \quad 5+\frac{5}{5}+5$$

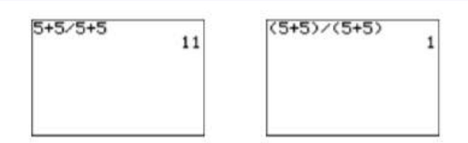
Let’s change the order of evaluation by using grouping symbols. Note that:

$$\begin{aligned}(5+5)/(5+5)&=10/10 && \text{Parentheses first.}\\ &=1 && \text{Divide: } 10/10=1\end{aligned}$$

That is:

$$(5+5)/(5+5) \quad \text{is equivalent to} \quad \frac{5+5}{5+5}$$

Enter $(5+5)(5+5)$ and press the ENTER key to produce the output shown in the second image in Figure $1.2.2$).



```
5+5/5+5
                 11
```

```
(5+5)/(5+5)
                  1
```

Figure** $1.2.2$**): Calculating $\frac{5+5}{5+5}$

Exercise $1.2.11$

Use the graphing calculator to evaluate:

$$\frac{10+10}{10+10}$$

Answer

1

The graphing calculator has memory locations available for “storing”values. They are lettered A-Z and appear on the calculator case, in alphabetic order as you move from left to right and down the keyboard. Storing values in these memory locations is an efficient way to evaluate algebraic expressions containing variables. Use the ALPHA key to access these memory locations.

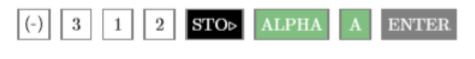
Figure** $1.2.3$**): Upper half of the TI-84.

Example 1.2.12

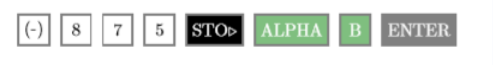
Use the graphing calculator to evaluate $|a|-|b|$ at $a=-312$ and $b=-875$.

Solution

First store -312 in the variable A with the following keystrokes. To select the letter A, press the ALPHA key, then the MATH key, located in the upper left-hand corner of the calculator (see Figure $1.2.3$).

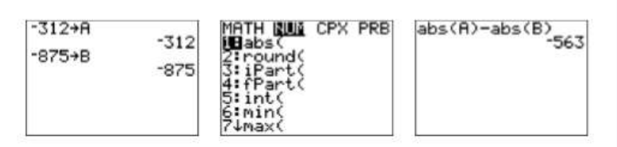


Next, store -875 in the variable B with the following keystrokes. To select the letter B, press the ALPHA key, then the APPS key.



The results of these keystrokes are shown in the first image in Figure $1.2.4$.

Now we need to enter the expression |a|−| b|. The absolute value function is located in the MATH menu. When you press the MATH key, you'll notice submenus MATH, NUM, CPX, and PRB across the top row of the MATH menu. Use the right-arrow key to select the NUM submenu (see the second image in Figure $1.2.4$). Note that abs(is the first entry on this menu. This is the absolute value function needed for this example. Enter the expression $\mathrm{abs(A)}-\mathrm{abs(B)}$ as shown in the third image in Figure $1.2.4$. Use the ALPHA key as described above to enter the variables A and B and close the parentheses using the right parentheses key from the keyboard. Press the ENTER key to evaluate your expression.



```
-312→A
            -312
-875→B
            -875
```

```
abs(A)-abs(B)
            -563
```

***Figure** $1.2.4$: Evaluate $|a|-|b|$ at $a=-312$ and $b=-875$.*

Thus, $|a|-|b|=-563$.

Exercise 1.2.12

Use the graphing calculator to evaluate $|a-b|$ at $a=-312$ and $b=-875$.

Answer

563

Contributors

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1.3: The Rational Numbers

We begin with the definition of a rational number.

Rational Numbers

Any number that can be expressed in the form p/q, where p and q are integers, $q \neq 0$, is called a rational number. The letter $\mathbb{Q}$ is used to represent the set of rational numbers. That is:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \text{ and } q \text{ are integers, } q \neq 0 \right\}$$

Because $-2/3$, $4/5$, and $123/(-12)$ have the form p/q, where p and q are integers, each is an example of a rational number. If you think you hear the word "fraction" when we say "rational number," you are correct in your thinking. Any number that can be expressed as a fraction, where the numerator and denominator are integers, is a rational number. Every integer is also a rational number. Take, for example, the integer -12. There are a number of ways we can express -12 as a fraction with integer numerator and denominator, $-12/1$, $24/(-2)$, and $-36/3$ being a few.

Reducing Fractions to Lowest Terms

First, we define what is meant by the *greatest common divisor* of two integers.

The Greatest Common Divisor

Given two integers a and b, the greatest common divisor of a and b is the largest integer that divides evenly (with no remainder) into both a and b. The notation $\mathrm{GCD}(a, b)$ is used to represent the greatest common divisor of a and b.

For example, $\mathrm{GCD}(12, 18) = 6$, $\mathrm{GCD}(32, 40) = 8$, and $\mathrm{GCD}(18, 27) = 9$.

We can now state when a fraction is reduced to lowest terms.

Lowest Terms

A fraction a/b is said to be reduced to lowest terms if and only if $\mathrm{GCD}(a, b) = 1$.

A common technique used to reduce a fraction to lowest terms is to divide both numerator and denominator by their greatest common divisor.

Example 1.3.1

Reduce $8/12$ to lowest terms.

Solution

Note that $\mathrm{GCD}(8, 12) = 4$. Divide both numerator and denominator by 4.

$$\begin{aligned} \frac{8}{12} &= \frac{8 \div 4}{12 \div 4} && \text{Divide numerator and denominator by } \mathrm{GCD}(8,12) = 4 \\ &= \frac{2}{3} && \text{Simplify numerator and denominator.} \\ \text{Thus, } 8/12 &= 2/3 \end{aligned}$$

Exercise 1.3.1

Reduce: $-48/60$.

Answer

$-4/5$

Recall the definition of a *prime number*.

Prime Number

A natural number greater than one is *prime* if and only if its only divisors are one and itself.

For example, 7 is prime (its only divisors are 1 and 7), but 14 is not (its divisors are 1, 2, 7, and 14). In like fashion, 2, 3, and 5 are prime, but 6, 15, and 21 are not prime.

Example $1.3.2$

Reduce $10/40$ to lowest terms.

Solution

Note that $\mathrm{GCD}(10, 40) = 10$. Divide numerator and denominator by 10.

$$\begin{aligned} \frac{10}{40} &= \frac{10 \div 10}{40 \div 10} && \text{Divide numerator and denominator by } \mathrm{GCD}(10,40)=10 \\ &= \frac{1}{4} && \text{Simplify numerator and denominator.} \\ \text{Thus, } 10/40 &= 1/4 \end{aligned}$$

Alternate Solution

Use *factor trees* to express both numerator and denominator as a product of prime factors.

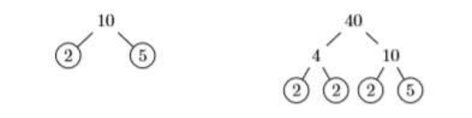


Figure $1.3.1$

Hence, $10 = 2 \cdot 5$ and $40 = 2 \cdot 2 \cdot 2 \cdot 5$. Now, to reduce $10/40$ to lowest terms, replace the numerator and denominator with their prime factorizations, then cancel factors that are in common to both numerator and denominator.

$$\begin{aligned} \frac{10}{40} &= \frac{2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5} && \text{Prime factor numerator and denominator.} \\ &= \frac{\cancel{2} \cdot \cancel{5}}{\cancel{2} \cdot 2 \cdot 2 \cdot \cancel{5}} && \text{Cancel common factors.} \\ &= \frac{1}{4} && \text{Simplify numerator and denominator.} \end{aligned} \tag{1.3.1}$$

When we cancel a 2 from both the numerator and denominator, we're actually dividing both numerator and denominator by 2. A similar statement can be made about canceling the 5. Canceling both 2 and a 5 is equivalent to dividing both numerator and denominator by 10. This explains the 1 in the numerator when all factors cancel.

Exercise $1.3.2$

Reduce $18/24$ to lowest terms. .

Answer

$3/4$

Example 1.3.2 demonstrates an important point.

When all factors cancel

When all of the factors cancel in either numerator or denominator, the resulting numerator or denominator is equal to one.

Multiplying Fractions

First, the definition.

Multiplication of Fractions

If a/b and c/d are two fractions, then their product is defined as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Thus, to find the product of a/b and c/d, simply multiply numerators and multiply denominators. For example:

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} \quad \text{and} \quad -\frac{2}{5} \cdot \frac{7}{3} = -\frac{14}{15} \quad \text{and} \quad -\frac{5}{8} \cdot \left(-\frac{1}{6}\right) = \frac{5}{48}$$

Like integer multiplication, like signs yield a positive answer, unlike signs yield a negative answer. Of course, when necessary, remember to reduce your answer to lowest terms.

Example 1.3.3

Simplify: $-\frac{14}{20} \cdot \frac{10}{21}$.

Solution

Multiply numerators and denominators, then reduce to lowest terms.

$$\begin{aligned} -\frac{14}{20} \cdot \frac{10}{21} &= -\frac{140}{420} && \text{Multiply numerators and denominators} \\ &= -\frac{2 \cdot 2 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 5 \cdot 7} && \text{Prime factor.} \\ &= -\frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{2} \cdot 3 \cdot \cancel{5} \cdot \cancel{7}} && \text{Cancel common factors.} \\ &= -\frac{1}{3} && \text{Simplify.} \end{aligned}$$

Note that when all the factors cancel from the numerator, you are left with a 1. Thus, $(-14/20) \cdot (10/21) = -1/3$.

Exercise 1.3.3

Simplify: $-\frac{8}{9} \cdot \left(-\frac{27}{20}\right)$.

Answer

$6/5$

Cancellation Rule

When multiplying fractions, cancel common factors according to the following rule: “Cancel a factor in a numerator for an identical factor in a denominator.”

The rule is “cancel something on the top for something on the bottom.” Thus, an alternate approach to multiplying fractions is to factor numerators and denominators in place, then cancel a factor in a numerator for an identical factor in a denominator.

Example 1.3.4

Simplify: $\frac{15}{8} \cdot \left(-\frac{14}{9}\right)$.

Solution

Factor numerators and denominators in place, then cancel common factors in the numerators for common factors in the denominators.

$$\begin{aligned} \frac{15}{8} \cdot \left(-\frac{14}{9}\right) &= \frac{3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \left(-\frac{2 \cdot 7}{3 \cdot 3}\right) && \text{Factor numerators and denominators.} \\ &= \frac{\cancel{3} \cdot 5}{\cancel{2} \cdot 2 \cdot 2} \cdot \left(-\frac{\cancel{2} \cdot 7}{\cancel{3} \cdot 3}\right) \\ & \qquad \text{Cancel a factor in a numerator for a common factor in a denominator.} \\ &= -\frac{35}{12} && \text{Multiply numerators and denominators.} \end{aligned}$$

Note that unlike signs yield a negative product. Thus, $(15/8) \cdot (-14/9) = -35/12$.

Exercise 1.3.4

Simplify: $-\frac{6}{45} \cdot \left(-\frac{35}{14}\right)$

Answer

$1/3$

Dividing Fractions

Every nonzero rational number has a *multiplicative inverse* or *reciprocal*.

The Reciprocal

If a is any nonzero rational number, then $1/a$ is called the multiplicative inverse or reciprocal of a, and:

$$a \cdot \frac{1}{a} = 1$$

Note that:

$$2 \cdot \frac{1}{2} = 1 \quad \text{and} \quad \frac{3}{5} \cdot \frac{5}{3} = 1 \quad \text{and} \quad -\frac{4}{7} \cdot \left(-\frac{7}{4}\right) = 1$$

Thus, the reciprocal of 2 is $1/2$, the reciprocal of $3/5$ is $5/3$, and the reciprocal of $-4/7$ is $-7/4$. Note that to find the reciprocal of a number, simply invert the number (flip it upside down). Now we can define the quotient of two fractions.

Division of Fractions

If a/b and c/d are two fractions, then their quotient is defined as follows:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

The above definition of division is summarized by the phrase “invert and multiply.”

Example 1.3.5

Simplify: $-\frac{35}{21} \div \left(-\frac{10}{12}\right)$.

Solution

Invert and multiply, then factor in place and cancel common factors in a numerator for common factors in a denominator.

$$\begin{aligned} -\frac{35}{21} \div \left(-\frac{10}{12}\right) &= -\frac{35}{21} \cdot \left(-\frac{12}{10}\right) && \text{Invert and multiply.} \\ &= -\frac{5 \cdot 7}{3 \cdot 7} \cdot \left(-\frac{2 \cdot 2 \cdot 3}{2 \cdot 5}\right) && \text{Prime factor.} \\ &= -\frac{\cancel{5} \cdot \cancel{7}}{\cancel{3} \cdot \cancel{7}} \cdot \left(-\frac{\cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{2} \cdot \cancel{5}}\right) && \text{Cancel common factors.} \\ &= \frac{2}{1} && \text{Multiply numerators and denominators.} \\ &= 2 && \text{Simplify.} \end{aligned}$$

Note that when all the factors in a denominator cancel, a 1 remains. Thus, $(-35/21) \div (-10/12) = 2$. Note also that like signs yield a positive result.

Exercise 1.3.5

Simplify: $-\frac{4}{9} \div \frac{27}{81}$.

Answer

$-4/3$

Adding Fractions

First the definition.

Addition of Fractions

If two fractions have a denominator in common, add the numerators and place the result over the common denominator. In symbols:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$

For example:

$$-\frac{3}{5}+\frac{7}{5}=\frac{4}{5} \quad \text{and} \quad -\frac{4}{3}+\left(-\frac{7}{3}\right)=-\frac{11}{3} \quad \text{and} \quad \frac{4}{7}+\left(-\frac{5}{7}\right)=-\frac{1}{7}$$

If the fractions do not posses a common denominator, first create equivalent fractions with a least common denominator, then add according to the rule above.

Least Common Denominator

If the fractions a/b and c/d do not share a common denominator, the least common denominator for b and d, written $\mathrm{LCD}(b, d)$, is defined as the smallest number divisible by both b and d.

Example 1.3.6

Simplify: $-\frac{3}{8}+\frac{5}{12}$.

Solution

The least common denominator in this case is the smallest number divisible by both 8 and 12. In this case, $\mathrm{LCD}(8,12)=24$. We first need to make equivalent fractions with a common denominator of 24.

$$\begin{aligned} -\frac{3}{8}+\frac{5}{12} &= -\frac{3}{8}\cdot\frac{3}{3}+\frac{5}{12}\cdot\frac{2}{2} && \text{Make equivalent fraction with a common denominator of 24} \\ &= -\frac{9}{24}+\frac{10}{24} && \text{Multiply numerators and denominators.} \\ &= \frac{1}{24} && \text{Add: } -9+10=1 \end{aligned}$$

Note how we add the numerators in the last step, placing the result over the common denominator. Thus, $-3/8+5/12=1/24$.

Exercise 1.3.6

Simplify: $-\frac{5}{6}+\frac{1}{9}$.

Answer

$-13/18$

Order of Operations

Rational numbers obey the same Rules Guiding Order of Operations as do the integers.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 1.3.7

Given $x=2/3$, $y=-3/5$, and $z=10/9$, evaluate $xy+yz$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variables in the expression $xy+yz$ with open parentheses. Next, substitute the given values of variables ($2/3$ for x, $-3/5$ for y, and $10/9$ for z) in the open parentheses.

$$\begin{aligned} xy+yz &= ()(\ \ )+(\ \ )(\ \ ) && \text{Replace variables with parentheses} \\ &= \left(\frac{2}{\ }\right)\left(-\frac{3}{\ }\right)+\left(-\frac{3}{\ }\right)\left(\frac{10}{\ }\right) && \text{Substitute: } 2/3 \text{ for } x, -3/5 \text{ for } y, \text{ and } 10/9 \text{ for } z \end{aligned}$$

Use the *Rules Guiding Order of Operations* to simplify.

$$\begin{aligned}
&= -\frac{6}{15} + \left(-\frac{30}{45}\right) && \text{Multiply} \\
&= -\frac{2}{5} + \left(-\frac{2}{3}\right) && \text{Reduce} \\
&= -\frac{2}{5} \cdot \frac{3}{3} + \left(-\frac{2}{3} \cdot \frac{5}{5}\right) && \text{Make equivalent fractions with a} \\
&= -\frac{6}{15} + \left(-\frac{10}{15}\right) && \text{Least Common Denominator} \\
&= -\frac{16}{15} && \text{Add}
\end{aligned}$$

Thus, if $x = 2/3, y = -3/5$, and $z = 10/9$, then $xy + yz = -16/15$

Exercise 1.3.7

Given $a = -1/2, b = 2/3$ and $c = -3/4$, evaluate the expression $a + bc$ and simplify the result.

Answer

-1

Example 1.3.8

Given $x = -3/5$, evaluate $-x^3$.

Solution

First, replace each occurrence of the variable x with open parentheses, then substitute $-3/5$ for x.

$$\begin{aligned}
-x^3 &= -()^3 && \text{Replace x with open parentheses.} \\
&= -\left(-\frac{3}{5}\right)^3 && \text{Substitute -3/5 for x} \\
&= -\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right) && \text{Write -3/5 as a factor three times} \\
&= -\left(-\frac{27}{125}\right) && \text{The product of three negative fractions is negative.} \\
&\text{Multiply numerators and denominators.} \\
&= \frac{27}{125} && \text{The opposite of -27/125 is 27/125}
\end{aligned}$$

Hence, $-x^3 = 27/125$, given $x = -3/5$.

Exercise 1.3.8

Simplify: $(-1/3)^4$.

Answer

$1/81$

Example 1.3.9

Given $a = -4/3$ and $b = -3/2$, evaluate $a^2 + 2ab - 3b^2$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variables in the expression $a^2 + 2ab - 3b^2$ with open parentheses.

Next, substitute the given values of variables ($-4/3$ for a and $-3/2$ for b) in the open parentheses.

$$\begin{aligned}
a^2 + 2ab - 3b^2 &= (\ )^2 + 2(\ )(\ ) - 3(\ )^2 \\
&= \left(-\frac{4}{3}\right)^2 + 2\left(-\frac{4}{3}\right)\left(-\frac{3}{2}\right) - 3\left(-\frac{3}{2}\right)^2
\end{aligned}$$

Next, evaluate the exponents: $(-4/3)^2 = 16/9$ and $(-3/2)^2 = 9/4$

$$= \frac{16}{9} + \frac{2}{1}\left(-\frac{4}{3}\right)\left(-\frac{3}{2}\right) - \frac{3}{1}\left(\frac{9}{4}\right)$$

Next, perform the multiplications and reduce.

$$\begin{aligned} &= \frac{16}{9} + \frac{24}{6} - \frac{27}{4} \\ &= \frac{16}{9} + 4 - \frac{27}{4} \end{aligned}$$

Make equivalent fractions with a common denominator, then add.

$$\begin{aligned} &= \frac{16}{9} \cdot \frac{4}{4} + 4 \cdot \frac{36}{36} - \frac{27}{4} \cdot \frac{9}{9} \\ &= \frac{64}{36} + \frac{144}{36} - \frac{243}{36} \\ &= -\frac{35}{36} \end{aligned}$$

Thus, if $a = -4/3$ and $b = -3/2$, then $a^2 + 2ab - 3b^2 = -35/36$

Exercise 1.3.9

Given $x = -3/4$ and $y = -4/5$,evaluate $x^2 - y^2$.

Answer

$-31/400$

Fractions on the Graphing Calculator

We must always remember that the graphing calculator is an “approximating machine.” In a small number of situations, it is capable of giving an exact answer, but for most calculations, the best we can hope for is an approximate answer.

However, the calculator gives accurate results for operations involving fractions, as long as we don’t use fractions with denominators that are too large for the calculator to respond with an exact answer.

Example 1.3.10

Use the graphing calculator to simplify each of the followingSimplify using the graphing calculator:

a. $\frac{2}{3} + \frac{1}{2}$
b. $\frac{2}{3} \cdot \frac{5}{7}$
c. $\frac{3}{5} \div \frac{1}{3}$

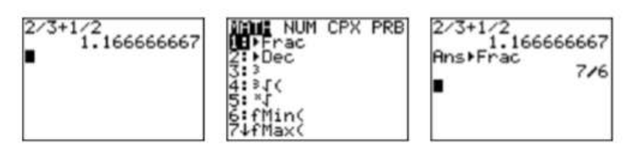
Solution

We enter each expression in turn.

a. The *Rules Guiding Order of Operations* tell us that we must perform divisions before additions. Thus, the expression $2/3 + 1/2$ is equivalent to:

$$\begin{aligned} 2/3 + 1/2 &= \frac{2}{3} + \frac{1}{2} \quad \text{Divide first.} \\ &= \frac{4}{6} + \frac{3}{6} \quad \text{Equivalent fractions with LCD.} \\ &= \frac{7}{6} \quad \text{Add.} \end{aligned}$$

Enter the expression $2/3 + 1/2$ on your calculator, then press the ENTER key. The result is shown in the first image in Figure 1.3.2. Next, press the MATH button, then select 1:Frac (see the second image in Figure 1.3.2) and press the ENTER key again. Note that the result shown in the third image in Figure 1.3.2 matches the correct answer of $7/6$ found above.



```
2/3+1/2
      1.166666667
```

```
2/3+1/2
      1.166666667
Ans▸Frac
              7/6
```

Figure $1.3.2$***:*** *Calculating* $2/3+1/2$.

b. The *Rules Guiding Order of Operations* tell us that there is no preference for division over multiplication, or vice-versa. We must perform divisions and multiplications as they occur, moving from left to right. Hence:

$$\begin{aligned} 2/3\times 5/7 &= \frac{2}{3}\times 5/7 && \text{Divide: } 2/3=\frac{2}{3} \\ &= \frac{10}{3}/7 && \text{Multiply: } \frac{2}{3}\times 5=\frac{10}{3} \\ &= \frac{10}{3}\times\frac{1}{7} && \text{Invert and multiply.} \\ &= \frac{10}{21} && \text{Multiply: } \frac{10}{3}\times\frac{1}{7}=\frac{10}{21} \end{aligned}$$

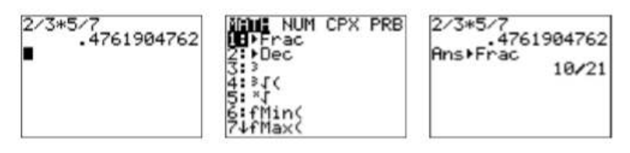
This is precisely the same result we get when we perform the following calculation.

$$\frac{2}{3}\times\frac{5}{7}=\frac{10}{21} \quad \text{Multiply numerators and denominators.}$$

Hence:

$$2/3\times 5/7 \quad \text{is equivalent to} \quad \frac{2}{3}\times\frac{5}{7}$$

Enter the expression $2/3\times 5/7$ on your calculator, then press the ENTER key. The result is shown in the first image in Figure $1.3.3$. Next, press the MATH button, then select 1:Frac (see the second image in Figure $1.3.3$) and press the ENTER key again. Note that the result shown in the third image in Figure $1.3.3$ matches the correct answer of $10/21$ found above.



```
2/3*5/7
      .4761904762
```

```
2/3*5/7
      .4761904762
Ans▸Frac
            10/21
```

Figure $1.3.3$***:*** *Calculating* $2/3\times 1/2$.

c. This example demonstrates that we need a constant reminder of the *Rules Guiding Order of Operations*. We know we need to invert and multiply in this situation.

$$\begin{aligned} \frac{3}{5}\div\frac{1}{3} &= \frac{3}{5}\times\frac{3}{1} && \text{Invert and multiply.} \\ &= \frac{9}{5} && \text{Multiply numerators and denominators.} \end{aligned}$$

So, the correct answer is 9/5. Enter the expression $3/5/1/3$ on your calculator, then press the ENTER key. Select 1:Frac from the MATH menu and press the ENTER key again. Note that the result in the first image in Figure $1.3.4$ does not match the correct answer of $9/5$ found above. What have we done wrong? If we follow the *Rules Guiding Order of Operations exactly*, then:

$$\begin{aligned} 3/5/1/3 &= \frac{3}{5}/1/3 && \text{Divide: } 3/5=\frac{3}{5} \\ &= \frac{3}{5}/3 && \text{Divide: } \frac{3}{5}/1=\frac{3}{5} \\ &= \frac{3}{5}\times\frac{1}{3} && \text{Invert and multiply.} \\ &= \frac{1}{5} && \text{Multiply: } \frac{3}{5}\times\frac{1}{3}=\frac{1}{5} \end{aligned}$$

This explains the answer found in the first image in Figure $1.3.4$. However, it also show that:

$$3/5/1/3 \quad \text{is not equivalent to} \quad \frac{3}{5} \div \frac{1}{3}$$

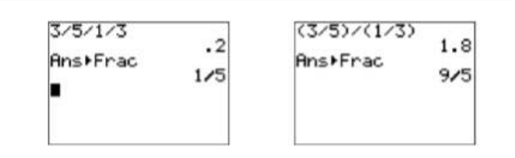
We can cure the problem by using grouping symbols.

$$\begin{aligned} (3/5)/(1/3) &= \frac{3}{5}/\frac{1}{3} && \color{red}{\text{Parentheses first.}} \\ &= \frac{3}{5} \div \frac{1}{3} && \color{red}{\text{is equivalent to } \div} \end{aligned}$$

Hence:

$$(3/5)/(1/3) \quad \text{is equivalent to} \quad \frac{3}{5} \div \frac{1}{3}$$

Enter the expression $(3/5)/(1/3)$ on your calculator, then press the ENTER key. Select 1:Frac from the MATH menu and press the ENTER key again. Note that the result in the second image in Figure $1.3.4$ matches the correct answer of $9/5$.



```
3/5/1/3
                 .2
Ans▸Frac
                1/5
▮
```

```
(3/5)/(1/3)
                1.8
Ans▸Frac
                9/5
```

***Figure** $1.3.4$**:** Calculating* $(3/5)/(1/3)$.

Exercise $1.3.10$

Simplify using the graphing calculator: $-\frac{4}{5}+\frac{8}{3}$.

Answer

$28/15$

Contributors

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1.4: Decimal Notation

Every rational number can be expressed using *decimal notation*. To change a fraction into its decimal equivalent, divide the numerator of the fraction by its denominator. In some cases the process will terminate, leaving a zero remainder. However, in other cases, the remainders will begin to repeat, providing a decimal representation that repeats itself in blocks.

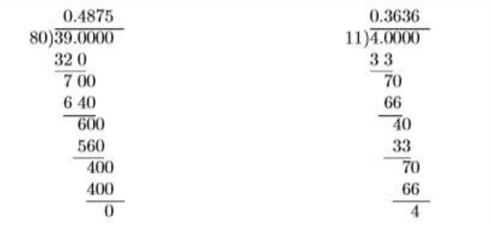
Example 1.4.1

Change each of the following fractions into decimals.

a. $\frac{39}{80}$
b. $\frac{4}{11}$

Solution

We perform two divisions, the one on the left to change $39/80$ to a decimal, the one on the right to find a decimal representation for $4/11$.



On the left, the division process terminates with a zero remainder. Hence, $39/80 = 0.4875$ is called a *terminating* decimal. On the right, the remainders repeat in a pattern and the quotient also repeats in blocks of two. Hence, $4/11 \; = 0.3636\ldots$ is called a repeating decimal. We can also use a repeating bar to write $4/11 \; = 0.\overline{36}$. The block under the repeating bar repeats itself indefinitely.

Exercise 1.4.1

Change $24/7$ to a decimal.

Answer

$3.\overline{428571}$

Vice-versa, any terminating decimal can be expressed as a fraction. You need only count the number of digits after the decimal point and use the same number of zeros in your denominator.

Example 1.4.2

Express each of the following decimals as fractions. Reduce your answers to lowest terms.

a. 0.055
b. 3.36

Solution

In each case, count the number of digits after the decimal point and include an equal number of zeros in the denominator.

a. In this example, there are three digits after the decimal point, so we place the number over 1000, which has three zeros after the one.

$$\begin{aligned} 0.055 &= \frac{55}{1000} \\ &= \frac{11}{200} \end{aligned}$$

b. In this example, there are two digits after the decimal point, so we place the number over 100, which has two zeros after the one.

336

$$\begin{aligned}3.36 &= \frac{}{100}\\ &= \frac{84}{25}\end{aligned}$$

Exercise 1.4.2

Change 0.45 to a fraction. Reduce to lowest terms.

Answer

$9/20$

As we saw in Example 1.4.1, the repeating decimal $0.\overline{36}$ is equivalent to the fraction $4/11$. Indeed, any repeating decimal can be written as a fraction. For example, $0.\overline{3} = 1/3$ and $0.\overline{142857} = 1/7$. In future courses you will learn a technique for changing any repeating decimal into an equivalent fraction. However, not all decimals terminate or repeat. For example, consider the decimal $0.42422422242222\ldots$, which neither terminates nor repeats. This number cannot be expressed using repeating bar notation because each iteration generates one additional 2. Because this number neither repeats nor terminates, it cannot be expressed as a fraction. Hence, $0.42422422242222\ldots$ is an example of an irrational number.

Irrational numbers

If a number cannot be expressed in the form p/q, where p and q are integers, $q \neq 0$, then the number is called an irrational number.

Real numbers

By including all of the rational and irrational numbers in one set, we form what is known as the set of real numbers.

The set of real numbers includes every single number we will use in this textbook and course.

Adding and Subtracting Decimals

When adding signed decimals, use the same rules you learned to use when adding signed integers or fractions.

Sign rules for addition

When adding two decimal numbers, use the following rules:

- To add two decimals with like signs, add their magnitudes and prefix their common sign.
- To add two decimals with unlike signs, subtract the smaller magnitude from the larger, then prefix the sign of the decimal number having the larger magnitude.

Example 1.4.3

Simplify:

a. $-2.3 + (-0.015)$
b. $-8.4 + 6.95$

Solution

a. In this problem, note that we have like signs. Hence, we add the magnitudes and prefix the common sign.

$$-2.3 + (-0.015) = -2.315$$

$$\begin{array}{r} 2.300 \\ +0.015 \\ \hline 2.315 \end{array}$$

b. In this problem, note that we have unlike signs. Thus, we first subtract the smaller magnitude from the larger magnitude, then prefix the sign of the decimal number with the larger magnitude.

$$-8.4 + 6.95 = -1.45$$

$$\begin{array}{r} 8.40 \\ -6.95 \end{array}$$

$$\begin{array}{r} \hline 1.45 \end{array}$$

Hence, $-2.3+(-0.015)=-2.315$ and $-8.4+6.95=-1.45$

Exercise 1.4.3

Simplify: $-22.6+18.47$.

Answer

-4.13

Subtraction still means “add the opposite.”

Example 1.4.4

Simplify:

a. $-5.6-8.4$
b. $-7.9-(-5.32)$

Solution

a. In this problem, first we add the oppposite. Then we note that we have like signs. Hence, we add the magnitudes and prefix the common sign.

$$\begin{aligned} -5.6-8.4 &= -5.6+(-8.4) \\ &= -14.0 \end{aligned}$$

$$\begin{array}{r} 5.6 \\ +8.4 \\ \hline 14.0 \end{array}$$

b. In this problem, first we add the opposite. Then we note that we have unlike signs. Thus, we first subtract the smaller magnitude from the larger magnitude, then prefix the sign of the decimal number with the larger magnitude.

$$\begin{aligned} -7.9-(-5.32) &= -7.9+5.32 \\ &= -2.58 \end{aligned}$$

$$\begin{array}{r} 7.90 \\ -5.32 \\ \hline 2.58 \end{array}$$

Hence, $-5.6-8.4=-14.0$ and $-7.9-(-5.32)=-2.58$.

Exercise 1.4.4

Simplify: $-22.6-18.47$.

Answer

-41.07

Multiplication and Division of Decimals

The sign rules for decimal multiplication and division are the same as the sign rules used for integers and fractions.

Sign Rules for multiplication and division

When multiplying or dividing two decimal numbers, use the following rules:

- Like signs give a positive result
- Unlike signs give a negative result

Multiplication of decimal numbers is fairly straightforward. First multiply the magnitudes of the numbers, ignoring the decimal points, then count the number of digits to the right of the decimal point in each factor. Place the decimal point in the product so that the number of digits to the right of the decimal points equals the sum of number of digits to the right of the decimal point in each factor.

Example 1.4.5

Simplify: $(-1.96)(2.8)$.

Solution

Multiply the magnitudes. The first decimal number has two digits to the right of the decimal point, the second has one digit to the right of the decimal point. Thus, we must place a total of three digits to the right of the decimal point in the product.

$$(-1.96)(2.8) = -5.488$$

$$\begin{array}{r} 1.96 \\ \times 2.8 \\ \hline 1568 \\ 392 \\ \hline 5.488 \end{array}$$

Note that unlike signs yield a negative product.

Exercise 1.4.5

Simplify: $(-12.5)(-23.4)$.

Answer

292.50

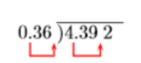
When dividing signed decimal numbers, ignore the signs and divide the magnitudes. Push the decimal point in the divisor to the end of the divisor, then move the decimal point in the dividend an equal number of spaces. This sets the decimal point in the quotient.

Example 1.4.6

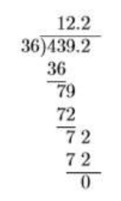
Simplify: $-4.392 \div (-0.36)$.

Solution

Divide the magnitudes. Move the decimal in the divisor to the end of the divisor. Move the decimal in the dividend an equal number of places (two places) to the right.



Place the decimal point in the quotient directly above the new position of the decimal point in the dividend, then divide.



Like signs yield a positive result. Hence, $-4.392 \div (-0.36) = 12.2$.

Exercise 1.4.6

Simplify: $-5.76/3.2$.

Answer

-1.8

Order of Operations

Decimal numbers obey the same *Rules Guiding Order of Operations* as do the integers and fractions.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

- Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
- Evaluate all exponents that appear in the expression.
- Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
- Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 1.4.7

Given $x = -0.12$, evaluate $-x^2$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variable x in the expression $-x^2$ with open parentheses. Next, substitute -0.12 for x in the open parentheses, then simplify.

$$\begin{aligned} -x^2 &= -(\quad)^2 && \text{Replace } x \text{ with open parentheses.} \\ &= -(-0.12)^2 && \text{Substitute } -0.12 \text{ for } x. \\ &= -(0.0144) && \text{Exponent: } (-0.12)^2 = 0.0144 \\ &= -0.0144 && \text{Negate.} \end{aligned}$$

Note that we square first, then we negate second. Thus, if $x = -0.12$, then $-x^2 = -0.0144$.

Exercise 1.4.7

Given $y = -0.2$, evaluate: $-y^4$.

Answer

-0.0016

Example 1.4.8

Given $x = -0.3$, evaluate $1.2x^2 - 3.4x$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variable x in the expression $1.2x^2 - 3.4x$ with open parentheses. Next, substitute -0.3 for x in the open parentheses, then simplify.

$$\begin{aligned} 1.2x^2 - 3.4x &= 1.2(\quad)^2 - 3.4(\quad) && \text{Replace } x \text{ with parentheses.} \\ &= 1.2(-0.3)^2 - 3.4(-0.3) && \text{Substitute } -0.3 \text{ for } x. \\ &= 1.2(0.09) - 3.4(-0.3) && \text{Exponent: } (-0.3)^2 = 0.09 \\ &= 0.108 - (-1.02) && \text{Multiply: } 1.2(0.09) = 0.108 \text{ and} \\ &= 0.108 + 1.02 && \text{Add the opposite.} \\ &= 1.128 && \text{Simplify.} \end{aligned}$$

Thus, if $x = -0.3$, then $1.2x^2 - 3.4x = 1.128$.

Exercise 1.4.8

Given $y = -0.15, \ \text{evaluate: } -1.4y^2 + 2.2y$.

Answer

-0.3615

We saw earlier that we can change a fraction to a decimal by dividing.

Example 1.4.9

Given $x = 2/5$, evaluate $-3.2x + 5$.

Solution

Following *Tips for Evaluating Algebraic Expressions*, first replace all occurrences of variable x in the expression $-3.2x + 5$ with open parentheses. Next, substitute $2/5$ for x in the open parentheses.

$$\begin{aligned} -3.2x + 5 &= -3.2(\ ) + 5 && \text{Replace } x \text{ with open parentheses.} \\ &= -3.2\left(\frac{2}{5}\right) + 5 && \text{Substitute } 2/5 \text{ for } x \end{aligned}$$

One approach is to change $2/5$ to a decimal by dividing the numerator by the denominator. Thus, $2/5 = 0.4$.

$$\begin{aligned} &= -3.2(0.4) + 5 && \text{Replace } 2/5 \text{ with } 0.4 \\ &= -1.28 + 5 && \text{Multiply: } -3.2(0.4) = -1.28 \\ &= 3.72 && \text{Add: } -1.28 + 5 = 3.72 \end{aligned}$$

Thus, if $x = 2/5$, then $-3.2x + 5 = 3.72$.

Exercise 1.4.9

Given $y = -3/4$, evaluate $-2.3y + 7$.

Answer

8.725

As we saw in Example 1.4.2, we can easily change a terminating decimal into a fraction by placing the number (without the decimal point) over the proper power of ten. The choice of the power of ten should match the number of digits to the right of the decimal point. For example:

$$0.411 = \frac{411}{1000} \quad \text{and} \quad 3.11 = \frac{311}{100} \quad \text{and} \quad 15.1111 = \frac{151111}{10000}$$

Note that the number of zeros in each denominator matches the number of digits to the right of the decimal point.

Example 1.4.10

Given $y = -0.25$, evaluate $-\frac{3}{5}y + 4$.

Solution

Following Tips for Evaluating Algebraic Expressions, first replace all occurrences of variable y in the expression $-(3/5)y + 4$ with open parentheses. Next, substitute -0.25 for y in the open parentheses.

$$\begin{aligned} -\frac{3}{5}y + 4 &= -\frac{3}{5}(\quad) + 4 && \text{Replace } y \text{ with open parentheses.} \\ &= -\frac{3}{5}(-0.25) + 4 && \text{Substitute } -0.25 \text{ for } y \end{aligned}$$

Place 25 over 100 to determine that $-0.25 = -25/100$, or after reduction, $-0.25 = -1/4$.

$$\begin{aligned} &= -\frac{3}{5}\left(-\frac{1}{4}\right) + 4 && \text{Replace } -0.25 \text{ with } -1/4 \\ &= \frac{3}{20} + 4 && \text{Multiply: } -\frac{3}{5}\left(-\frac{1}{4}\right) = \frac{3}{20} \\ &= \frac{3}{20} + \frac{80}{20} && \text{Make equivalent fractions with LCD.} \end{aligned}$$

$$= \frac{83}{20} \quad \text{Add.}$$

Thus, if $y = -0.25$, then $-(3/5)y + 4 = 83/20$.

Exercise 1.4.10

Given $z = -0.4$, evaluate: $5 - \frac{4}{5}z$.

Answer

$133/25$

Rounding Using the Graphing Calculator

Here is the algorithm for rounding a decimal number to a particular place.

Rules for rounding

To round a number to a particular place, follow these steps:

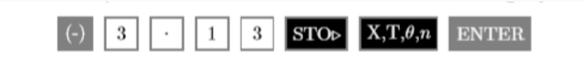
1. Mark the place you wish to round to. The digit in this place is called the rounding digit.
2. Mark the digit in the place to the immediate right of the rounding digit. This is called the test digit.
   a. If the test digit is greater than or equal to 5, add 1 to the rounding digit, then replace all digits to the right of the rounding digit with zeros. Trailing zeros to the right of the decimal point may be deleted.
   b. If the test digit is less than 5, keep the rounding digit the same, then replace all digits to the right of the rounding digit with zeros. Trailing zeros to the right of the decimal point may be deleted.

Example 1.4.11

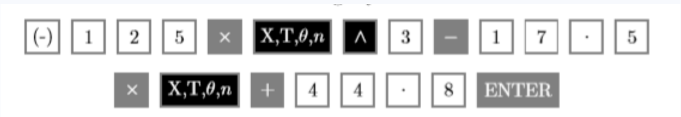
Use your graphing calculator to evaluate $125x^3 - 17.5x + 44.8$ at $x = -3.13$. Round your answer to the nearest tenth.

Solution

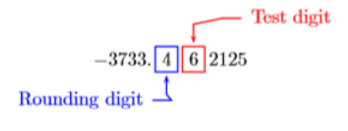
First, store -3.13 in the variable X with the following keystrokes.

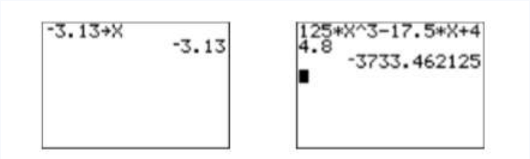


The result is shown in the first image in Figure $1.4.1$. Next, enter the expression $125x^3 - 17.5x + 44.8$ with the following keystrokes.



The result is shown in the second image in Figure $1.4.1$. Thus, the answer is approximately -3733.462125. We now need to round this answer to the nearest tenth. Mark the rounding digit in the tenths place and the test digit to its immediate right.





```
-3.13→X
          -3.13
```

```
125*X^3-17.5*X+4
4.8
   -3733.462125
```

***Figure** $1.4.1$**:** Evaluate* $125x^3 - 17.5x + 44.8$ *at* $x = -3.13$

Because the test digit is greater than or equal to 5, add 1 to the rounding digit, then replace all digits to the right of the rounding digit with zeros.

$$-3733.462125 \approx -3733.500000$$

Delete the trailing zeros from end of the fractional part of a decimal. This does not change our answer’s value.

$$-3733.462125 \approx -3733.5$$

Therefore, if $x = -3.13$. then to the nearest tenth:

$$125x^3 - 17.5x + 44.8 \approx -3733.5$$

Exercise 1.4.11

Evaluate $x^3 - 3x$ at $x = -1.012$. Round to the nearest hundredth.

Answer

2.0

Contributors

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1.5: Algebraic Expressions

The associative property of multiplication is valid for all numbers.

Associative Property of Multiplication

Let a, b, and c be any numbers. Then:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

The associative property of multiplication is useful in a number of situations.

Example 1.5.1

Simplify: $-3(4y)$.

Solution

Currently, the grouping $-3(4y)$ demands that we first multiply 4 and y. However, we can use the associative property of multiplication to regroup, first multiplying -3 and 4.

$$\begin{aligned} -3(4y) &= (-3 \cdot 4)y && \text{The associative property of multiplication.} \\ &= -12y && \text{Multiply: } -3 \cdot 4 = -12 \end{aligned}$$

Thus, $-3(4y) = -12y$.

Exercise 1.5.1

Simplify: $2(3x)$.

Answer

$6x$

Let's look at another example.

Example 1.5.2

Simplify: $-2(-4xy)$.

Solution

Currently, the grouping $-2(-4xy)$ demands that we first multiply -4 and xy. However, we can use the associative property of multiplication to regroup, first multiplying -2 and -4.

$$\begin{aligned} -2(-4xy) &= (-2 \cdot (-4))xy && \text{The associative property of multiplication.} \\ &= 8xy && \text{Multiply: } -2 \cdot (-4) = 8 \end{aligned}$$

Thus, $-2(-4xy) = 8xy$.

Exercise 1.5.2

Simplify: $-3\left(-8u^2\right)$.

Answer

$24u^2$

In practice, we can move quicker if we perform the regrouping mentally, then simply write down the answer. For example:

$$-2(-4t) = 8t \quad \text{and} \quad 2\left(-5z^2\right) = -10z^2 \quad \text{and} \quad -3\left(4u^3\right) = -12u^3$$

The Distributive Property

We now discuss a property that couples addition and multiplication. Consider the expression $2 \cdot (3+5)$. The *Rules Guiding Order of Operations* require that we first simplify the expression inside the parentheses.

$$2 \cdot (3+5) = 2 \cdot 8 \qquad \text{Add: } 3+5=8$$

$$= 16 \quad \text{Multiply: } 2 \cdot 8 = 16$$

Alternatively, we can instead distribute the 2 times each term in the parentheses. That is, we will first multiply the 3 by 2, then multiply the 5 by 2. Then we add the results.

$$\begin{aligned} 2 \cdot (3+5) &= 2 \cdot 3 + 2 \cdot 5 \quad \text{Distribute the 2.} \\ &= 6 + 10 \quad \text{Multiply: } 2 \cdot 3 = 6 \text{ and } 2 \cdot 5 = 10 \\ &= 16 \quad \text{Add: } 6 + 10 = 16 \end{aligned}$$

Note that both methods produce the same result, namely 16. This example demonstrates an extremely important property of numbers called the *distributive property*.

The Distributive Property

Let a, b, and c be any numbers. Then:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

That is, multiplication is distributive with respect to addition.

Example 1.5.3

Use the distributive property to expand $2(3x+7)$.

Solution

First distribute the 2 times each term in the parentheses. Then simplify.

$$\begin{aligned} 2(3x+7) &= 2(3x) + 2(7) \quad \text{Use the distributive property.} \\ &= 6x + 14 \quad \text{Multiply: } 2(3x) = 6x \text{ and } 2(7) = 14 \end{aligned}$$

Thus, $2(3x+7) = 6x + 14$.

Exercise 1.5.3

Expand: $5(2y+7)$.

Answer

$10y + 35$

Multiplication is also distributive with respect to subtraction.

Example 1.5.4

Use the distributive property to expand $-2(5y-6)$.

Solution

Change to addition by adding the opposite, then apply the distributive property.

$$\begin{aligned} -2(5y-6) &= -2(5y + (-6)) \quad \text{Add the opposite.} \\ &= -2(5y) + (-2)(-6) \quad \text{Use the distributive property.} \\ &= -10y + 12 \quad \text{Multiply: } -2(5y) = -10y \text{ and } (-2)(-6) = 12 \end{aligned}$$

Thus, $-2(5y-6) = -10y + 12$

Exercise 1.5.4

Expand: $-3(2z-7)$.

Answer

$-6z + 21$

Speeding Things Up a Bit

In Example 1.5.4, we changed the subtraction to addition, applied the distributive property, then several steps later we were finished. However, if you understand that subtraction is really the same as adding the opposite, and if you are willing to do a few steps in your head, you should be able to simply write down the answer immediately following the given problem.

If you look at the expression $-2(5y-6)$ from Example 1.5.4 again, only this time think "multiply -2 times $5y$, then multiply -2 times -6, then the result is immediate.

$$-2(5y-6)=-10y+12$$

Let's try this "speeding it up" technique in a couple more examples.

Example 1.5.5

Use the distributive property to expand $-3(-2x+5y-12)$.

Solution

To distribute the -3, we simply think as follows: "$-3(-2x)=6x$, $-3(5y)=-15y$, and $-3(-12)=36$." This sort of thinking allows us to write down the answer immediately without any additional steps.

$$-3(-2x+5y-12)=6x-15y+36$$

Exercise 1.5.5

Expand: $-3(-2a+3b-7)$.

Answer

$6a-9b+21$

Example 1.5.6

Use the distributive property to expand $-5(-2a-5b+8)$.

Solution

To distribute the -5, we simply think as follows: "$-5(-2a)=10a$, $-5(-5b)=25b$, and $-5(8)=-40$." This sort of thinking allows us to write down the answer immediately without any additional steps.

$$-5(-2a-5b+8)=10a+25b-40$$

Exercise 1.5.6

Expand: $-4(-x-2y-7)$.

Answer

$4x+8y+28$

Distributing a Negative Sign

Recall that negating a number is equivalent to multiplying the number by -1.

Multiplicative Property of Minus One

If a is any number, then:

$$(-1)a=-a$$

This means that if we negate an expression, it is equivalent to multiplying the expression by -1.

Example 1.5.7

Expand $-(7x-8y-10)$.

Solution

First, negating is equivalent to multiplying by -1. Then we can change subtraction to addition by "adding the opposite" and use the distributive property to finish the expansion.

$$\begin{aligned} -(7x-8y-10) &= -1(7x-8y-10) \quad \text{Negating is equivalent to multiplying by } -1 \\ &= -1(7x+(-8y)+(-10)) \quad \text{Add the opposite.} \\ &= -1(7x)+(-1)(-8y)+(-1)(-10) \quad \text{Distribute the } -1 \\ &= -7x+8y+10 \quad \text{Multiply.} \end{aligned}$$

Thus, $-(7x-8y-10) = -7x+8y+10$

Exercise 1.5.7

Expand: $-(-a-2b+11)$.

Answer

$a+2b-11$

While being mathematically precise, the technique of Example 1.5.7 can be simplified by noting that negating an expression surrounded by parentheses simply changes the sign of each term inside the parentheses to the opposite sign.

Once we understand this, we can simply "distribute the minus sign" and write:

$$-(7x-8y-10) = -7x+8y+10$$

In similar fashion,

$$-(-3a+5b-c) = 3a-5b+c$$

and,

$$-(-3x-8y+11) = 3x+8y-11$$

Combining Like Terms

We can use the distributive property to distribute a number times a sum.

$$a(b+c) = ab+ac$$

However, the distributive property can also be used in reverse, to "unmultiply" or factor an expression. Thus, we can start with the expression $ab+ac$ and "factor out" the common factor a as follows:

$$ab+ac = a(b+c)$$

You can also factor out the common factor on the right.

$$ac+bc = (a+b)c$$

We can use this latter technique to combine *like terms*.

Example 1.5.8

Simplify: $7x+5x$.

Solution

Use the distributive property to factor out the common factor x from each term, then simplify the result.

$$\begin{aligned} 7x+5x &= (7+5)x \quad \text{Factor out an } x \text{ using the distributive property.} \\ &= 12x \quad \text{Simplify: } 7+5=12 \end{aligned}$$

Thus, $7x+5x = 12x$.

Exercise 1.5.8

Simplify: $3y+8y$.

Answer

$11y$

Example 1.5.9

Simplify: $-8a^2+5a^2$.

Solution

Use the distributive property to factor out the common factor a^2 from each term, then simplify the result.

$$\begin{aligned} -8a^2+5a^2 &= (-8+5)a^2 && \text{Factor out an } a^2 \text{ using the distributive property.} \\ &= -3a^2 && \text{Simplify: } -8+5=-3 \end{aligned}$$

Thus, $-8a^2+5a^2=-3a^2$.

Exercise 1.5.9

Simplify: $-5z^3+9z^3$.

Answer

$4z^3$

Examples 1.5.8 and 1.5.9 combine what are known as “like terms.” Examples 1.5.8 and 1.5.9 also suggest a possible shortcut for combining like terms.

Like Terms

Two terms are called like terms if they have identical variable parts, which means that the terms must contain the same variables raised to the same exponents.

For example, $2x^2y$ and $11x^2y$ are like terms because they contain identical variables raised to the same exponents. On the other hand, $-3st^2$ and $4s^2t$ are not like terms. They contain the same variables, but the variables are not raised to the same exponents.

Consider the like terms $2x^2y$ and $11x^2y$. The numbers 2 and 11 are called the coefficients of the like terms. We can use the distributive property to combine these like terms as we did in Examples 1.5.8 and 1.5.9, factoring out the common factor x^2y.

$$\begin{aligned} 2x^2y+11x^2y &= (2+11)x^2y \\ &= 13x^2y \end{aligned}$$

However, a much quicker approach is simply to add the coefficients of the like terms, keeping the same variable part. That is, $2+11=13$, so:

$$2x^2y+11x^2y=13x^2y$$

This is the procedure we will follow from now on.

Example 1.5.10

Simplify: $-8w^2+17w^2$.

Solution

These are like terms. If we add the coefficients -8 and 17, we get 9. Thus:

$$-8w^2+17w^2=9w^2 \quad \text{Add the coefficients and repeat the variable part.}$$

Exercise 1.5.10

Simplify: $4ab-15ab$.

Answer

$-11ab$

Example 1.5.11

Simplify: $-4uv - 9uv$.

Solution

These are like terms. If we add -4 and -9, we get -13. Thus:

$$-4uv - 9uv = -13uv \quad \text{Add the coefficients and repeat the variable part.}$$

Exercise 1.5.11

Simplify: $-3xy - 8xy$.

Answer

$-11xy$

Example 1.5.12

Simplify: $-3x^2y + 2xy^2$

Solution

These are not like terms. They do not have the same variable parts. They do have the same variables, but the variables are not raised to the same exponents. Consequently, this expression is already simplified as much as possible.

$$-3x^2y + 2xy^2 \quad \text{Unlike terms. Already simplified.}$$

Exercise 1.5.12

Simplify: $5ab + 11bc$.

Answer

$5ab + 11bc$

Sometimes we have more than just a single pair of like terms. In that case, we want to group together the like terms and combine them.

Example 1.5.13

Simplify: $-8u - 4v - 12u + 9v$.

Solution

Use the associative and commutative property of addition to change the order and regroup, then combine line terms.

$$\begin{aligned} -8u - 4v - 12u + 9v &= (-8u - 12u) + (-4v + 9v) && \text{Reorder and regroup.} \\ &= -20u + 5v && \text{Combine like terms.} \end{aligned}$$

Note that $-8u - 12u = -20u$ and $-4v + 9v = 5v$.

Alternate solution

You may skip the reordering and regrouping step if you wish, simply combining like terms mentally. That is, it is entirely possible to order your work as follows:

$$-8u - 4v - 12u + 9v = -20u + 5v \quad \text{Combine like terms.}$$

Exercise 1.5.13

Simplify: $-3z^2 + 4z - 8z^2 - 9z$.

Answer

$-11z^2 - 5z$

In Example 1.5.13, the "Alternate solution" allows us to move more quickly and will be the technique we follow from here on, grouping and combining terms mentally.

Order of Operations

Now that we know how to combine like terms, let's tackle some more complicated expressions that require the *Rules Guiding Order of Operations*.

Rules Guiding Order of Operations

When evaluating expressions, proceed in the following order.

1. Evaluate expressions contained in grouping symbols first. If grouping symbols are nested, evaluate the expression in the innermost pair of grouping symbols first.
2. Evaluate all exponents that appear in the expression.
3. Perform all multiplications and divisions in the order that they appear in the expression, moving left to right.
4. Perform all additions and subtractions in the order that they appear in the expression, moving left to right.

Example 1.5.14

Simplify: $4(-3a+2b)-3(4a-5b)$.

Solution

Use the distributive property to distribute the 4 and the -3, then combine like terms.

$$\begin{aligned} 4(-3a+2b)-3(4a-5b) &= -12a+8b-12a+15b \quad \color{red}{\text{Distribute.}} \\ &= -24a+23b \quad \color{red}{\text{Combine like terms.}} \end{aligned}$$

Note that $-12a-12a=-24a$ and $8b+15b=23b$

Exercise 1.5.14

Simplify: $-2x-3(5-2x)$.

Answer

$4x-15$

Example 1.5.15

Simplify: $-2(3x-4y)-(5x-2y)$.

Solution

Use the distributive property to multiply -2 times $3x-4y$, then distribute the minus sign times each term of the expression $5x-2y$. After that, combine like terms.

$$\begin{aligned} -2(3x-4y)-(5x-2y) &= -6x+8y-5x+2y \quad \color{red}{\text{Distribute.}} \\ &= -11x+10y \quad \color{red}{\text{Combine like terms.}} \end{aligned}$$

Note that $-6x-5x=-11x$ and $8y+2y=10y$.

Exercise 1.5.15

Simplify: $-3(u+v)-(u-5v)$.

Answer

$-4u+2v$

Example 1.5.16

Simplify: $-2\left(x^2y-3xy^2\right)-4\left(-x^2y+3xy^2\right)$.

Solution

Use the distributive property to multiply -2 times x^2y-3xy^2 and -4 times $-x^2y+3xy^2$. After that, combine like terms.

$$\begin{aligned} -2\left(x^2y-3xy^2\right)-4\left(-x^2y+3xy^2\right) &= -2x^2y+6xy^2+4x^2y-12xy^2 \\ &= 2x^2y-6xy^2 \end{aligned}$$

Note that $-2x^2y+4x^2y=2x^2y$ and $6xy^2-12xy^2=-6xy^2$.

Exercise 1.5.16

Simplify: $8u^2v-3\left(u^2v+4uv^2\right)$.

Answer

$5u^2v-12uv^2$

When grouping symbols are nested, evaluate the expression inside the innermost pair of grouping symbols first.

Example 1.5.17

Simplify: $-2x-2(-2x-2[-2x-2])$.

Solution

Inside the parentheses, we have the expression $-2x-2[-2x-2]$. *The Rules Guiding Order of Operations* dictate that we should multiply first, expanding $-2[-2x-2]$ and combining like terms.

$$\begin{aligned} -2x-2({\color{red}-2x-2[-2x-2]}) &= -2x-2({\color{red}-2x+4x+2}) \\ &= -2x-2({\color{red}2x+2}) \end{aligned}$$

In the remaining expression, we again multiply first, expanding $-2(2x+2)$ and combining like terms.

$$\begin{aligned} &= -2x-4x-4 \\ &= -6x-4 \end{aligned}$$

Exercise 1.5.17

Simplify: $x-2[-x+4(x+1)]$.

Answer

$-5x-8$

Contributors

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1.E: The Arithmetic of Numbers (Exercises)

1.1 An Introduction to the Integers

In Exercises 1-8, simplify each of the following expressions.

1) $|5|$

Answer

5

2) $|1|$

3) $|-2|$

Answer

2

4) $|-1|$

5) $|2|$

Answer

2

6) $|8|$

7) $|-4|$

Answer

4

8) $|-6|$

In Exercises 9-24, simplify each of the following expressions as much as possible.

9) $-91+(-147)$

Answer

-238

10) $-23+(-13)$

11) $96+145$

Answer

241

12) $16+127$

13) $-76+46$

Answer

-30

14) $-11+21$

15) $-59+(-12)$

Answer

-71

16) $-40+(-58)$

17) $37+(-86)$

Answer

-49

18) $143+(-88)$

19) $66+(-85)$

Answer

-19

20) $33+(-41)$

21) $57+20$

Answer

77

22) $66+110$

23) $-48+127$

Answer

79

24) $-48+92$

In Exercises 25-32, find the difference.

25) $-20-(-10)$

Answer

-10

26) $-20-(-20)$

27) $-62-7$

Answer

-69

28) $-82-62$

29) $-77-26$

Answer

-103

30) $-96-92$

31) $-7-(-16)$

Answer

9

32) $-20-(-5)$

In Exercises 33-40, compute the exact value.

33) $(-8)^6$

Answer

262144

34) $(-3)^5$

35) $(-7)^5$

Answer

-16807

36) $(-4)^6$

37) $(-9)^2$

Answer

81

38) $(-4)^2$

39) $(-4)^4$

Answer

256

40) $(-5)^4$

In Exercises 41-52, use your graphing calculator to compute the given expression.

41) $-562-1728$

Answer

-2290

42) $-3125-(-576)$

43) $-400-(-8225)$

Answer

7825

44) $-8176+578$

45) $(-856)(232)$

Answer

-198592

46) $(-335)(-87)$

47) $(-815)(-3579)$

Answer

2916885

48) $(753)(-9753)$

49) $(-18)^3$

Answer

-5832

50) $(-16)^4$

51) $(-13)^5$

Answer

-371293

52) $(-15)^6$

1.2 Order of Operations

In Exercises 1-18, simplify the given expression.

1) $-12+6(-4)$

Answer

-36

2) $11+11(7)$

3) $-(-2)^5$

Answer

32

4) $-(-5)^3$

5) $-|-40|$

Answer

-40

6) $-|-42|$

7) $-24/(-6)(-1)$

Answer

-4

8) 45$/(-3)(3)$

9) $-(-50)$

Answer

50

10) $-(-30)$

11) -3^5

Answer

-243

12) -3^2

13) $48 \div 4(6)$

Answer

72

14) $96 \div 6(4)$

15) $-52 - 8(-8)$

Answer

12

16) $-8 - 7(-3)$

17) $(-2)^4$

Answer

16

18) $(-4)^4$

In Exercises 19-42, simplify the given expression.

19) $9 - 3(2)^2$

Answer

-3

20) $-4 - 4(2)^2$

21) $17 - 10|13 - 14|$

Answer

7

22) $18 - 3| - 20 - 5|$

23) $-4 + 5(-4)^3$

Answer

-324

24) $3 + 3(-4)^3$

25) $8 + 5(-1 - 6)$

Answer

-27

26) $8 + 4(-5 - 5)$

27) $(10 - 8)^2 - (7 - 5)^3$

Answer

-4

28) $(8 - 10)^2 - (4 - 5)^3$

29) $6-9(6-4(9-7))$

Answer

24

30) $4-3(3-5(7-2))$

31) $-6-5(4-6)$

Answer

4

32) $-5-5(-7-7)$

33) $9+(9-6)^3-5$

Answer

31

34) $12+(8-3)^3-6$

35) $-5+3(4)^2$

Answer

43

36) $2+3(2)^2$

37) $8-(5-2)^3+6$

Answer

-13

38) $9-(12-11)^2+4$

39) $|6-15|-|-17-11|$

Answer

-19

40) $|-18-19|-|-3-12|$

41) $5-5(5-6(6-4))$

Answer

40

42) $4-6(4-7(8-5))$

In Exercises 43-58, evaluate the expression at the given values of x and y.

43) $4x^2+3xy+4y^2$ at $x=-3$ and $y=0$

Answer

36

44) $3x^2-3xy+2y^2$ at $x=4$ and $y=-3$

45) $-8x+9$ at $x=-9$

Answer

81

46) $-12x+10$ at $x=2$

47) $-5x^2+2xy-4y^2$ at $x=5$ and $y=0$

Answer

-125

48) $3x^2+3xy-5y^2$ at $x=0$ and $y=3$

49) $3x^2+3x-4$ at $x=5$

Answer

86

50) $2x^2+6x-5$ at $x=6$

51) $-2x^2+2y^2$ at $x=1$ and $y=-2$

Answer

6

52) $-5x^2+5y^2$ at $x=-4$ and $y=0$

53) $-3x^2-6x+3$ at $x=2$

Answer

-21

54) $-7x^2+9x+5$ at $x=-7$

55) $-6x-1$ at $x=1$

Answer

-7

56) $10x+7$ at $x=9$

57) $3x^2-2y^2$ at $x=-3$ and $y=-2$

Answer

19

58) $-3x^2+2y^2$ at $x=2$ and $y=2$

59) Evaluate $\dfrac{a^2+b^2}{a+b}$ at $a=27$ and $b=-30$.

Answer

-543

60) Evaluate $\dfrac{a^2+b^2}{a+b}$ at $a=-63$ and $b=77$.

61) Evaluate $\dfrac{a+b}{c-d}$ at $a=-42$, $b=25$, $c=26$, and $d=43$.

Answer

1

62) Evaluate $\frac{a+b}{c-d}$ at $a=38$, $b=42$, $c=10$, and $d=50$.

63) Evaluate $\frac{a-b}{cd}$ at $a=-7$, $b=48$, $c=5$, and $d=11$.

Answer

-1

64) Evaluate $\frac{a-b}{cd}$ at $a=-46$, $b=46$, $c=23$, and $d=2$.

65) Evaluate the expressions a^2+b^2 and $(a+b)^2$ at $a=3$ and $b=4$. Do the expressions produce the same results?

Answer

No

66) Evaluate the expressions a^2b^2 and $(ab)^2$ at $a=3$ and $b=4$. Do the expressions produce the same results?

67) Evaluate the expressions $|a||b|$ and $|ab|$ at $a=-3$ and $b=5$. Do the expressions produce the same results?

Answer

Yes

68) Evaluate the expressions $|a|+|b|$ and $|a+b|$ at $a=-3$ and $b=5$. Do the expressions produce the same results?

In Exercises 69-72, use a graphing calculator to evaluate the given expression.

69) $-236-324(-576+57)$

Answer

167920

70) $-443+27(-414-22)$

71) $\frac{270-900}{300-174}$

Answer

-5

72) $\frac{3000-952}{144-400}$

73) Use a graphing calculator to evaluate the expression $\frac{a^2+b^2}{a+b}$ at $a=-93$ and $b=84$ by first storing -93 in the variable A and 84 in the variable B, then entering the expression $(A^2+B^2)/(A+B)$.

Answer

-1745

74) Use a graphing calculator to evaluate the expression $\frac{a^2+b^2}{a+b}$ at $a=-76$ and $b=77$ by first storing -76 in the variable A and 77 in the variable B, then entering the expression $(A^2+B^2)/(A+B)$.

75) The formula $F=\frac{9}{5}C+32$ will change a Celsius temperature to a Fahrenheit temperature. Given that the Celsius temperature is $C=60^{\circ}\mathrm{C}$, find the equivalent Fahrenheit temperature.

Answer

$140^{\circ}\mathrm{F}$

76) The surface area of a cardboard box is given by the formula

$$S=2WH+2LH+2LW$$

where W and L are the width and length of the base of the box and H is its height. If $W=2$ centimeters, $L=8$ centimeters, and $H=2$ centimeters, find the surface area of the box.

77) The kinetic energy (in joules) of an object having mass m (in kilograms) and velocity v (in meters per second) is given by the formula $K=\frac{1}{2}mv^2$. Given that the mass of the object is $m=7$ kilograms and its velocity is $v=50$ meters per second, calculate the kinetic energy of the object.

Answer

8750 joules

78) The area of a trapezoid is given by the formula $A=\frac{1}{2}\left(b_1+b_2\right)h$, where b_1 and b_2 are the lengths of the parallel bases and h is the height of the trapezoid. If the lengths of the bases are 21 yards and 11 yards, respectively, and if the height is 22 yards, find the area of the trapezoid.

1.3 The Rational Numbers

In Exercises 1-6, reduce the given fraction to lowest terms by dividing numerator and denominator by the their greatest common divisor.

1) $\frac{20}{50}$

Answer

$\frac{2}{5}$

2) $\frac{36}{38}$

3) $\frac{10}{48}$

Answer

$\frac{5}{24}$

4) $\frac{36}{14}$

5) $\frac{24}{45}$

Answer

$\frac{8}{15}$

6) $\frac{21}{36}$

In Exercises 7-12, reduce the given fraction to lowest terms by prime factoring both numerator and denominator and canceling common factors.

7) $\frac{153}{170}$

Answer:

$$\frac{9}{10}$$

8) $\frac{198}{144}$

In Exercises 9-24, simplify each of the following expressions as much as possible.

9) $\frac{188}{141}$

Answer

$\frac{4}{3}$

10) $\frac{171}{144}$

11) $\frac{159}{106}$

Answer

$\frac{3}{2}$

12) $\frac{140}{133}$

In Exercises 13-18, for each of the following problems, multiply numerators and denominators, then prime factor and cancel to reduce your answer to lowest terms.

13) $\frac{20}{8} \cdot \left(-\frac{18}{13}\right)$

Answer

$-\frac{45}{13}$

14) $\frac{18}{16} \cdot \left(-\frac{2}{5}\right)$

15) $-\frac{19}{4} \cdot \left(-\frac{18}{13}\right)$

Answer

$\frac{171}{26}$

16) $-\frac{3}{2} \cdot \left(-\frac{14}{6}\right)$

17) $-\frac{16}{8} \cdot \frac{19}{6}$

Answer

$-\frac{19}{3}$

18) $-\frac{14}{4} \cdot \frac{7}{17}$

In Exercises 19-24, for each of the following problems, first prime factor all numerators and denominators, then cancel. After canceling, multiply numerators and denominators.

19) $-\frac{5}{6} \cdot \left(-\frac{12}{49}\right)$

Answer

$\frac{10}{49}$

20) $-\frac{36}{17} \cdot \left(-\frac{21}{46}\right)$

21) $-\frac{21}{10} \cdot \frac{12}{55}$

Answer

$-\frac{126}{275}$

22) $-\frac{49}{13} \cdot \frac{52}{51}$

23) $\frac{55}{29} \cdot \left(-\frac{54}{11}\right)$

Answer

$-\frac{270}{29}$

24) $\frac{7}{13} \cdot \left(-\frac{55}{49}\right)$

In Exercises 25-30, divide. Be sure your answer is reduced to lowest terms.

25) $\frac{50}{39} \div \left(-\frac{5}{58}\right)$

Answer

$-\frac{580}{39}$

26) $\frac{31}{25} \div \left(-\frac{4}{5}\right)$

27) $-\frac{60}{17} \div \frac{34}{31}$

Answer

$-\frac{930}{289}$

28) $-\frac{27}{28} \div \frac{45}{23}$

29) $-\frac{7}{10} \div \left(-\frac{13}{28}\right)$

Answer

$\frac{98}{65}$

30) $-\frac{4}{13} \div \left(-\frac{48}{35}\right)$

In Exercises 31-38, add or subtract the fractions, as indicated, and simplify your result.

31) $-\frac{5}{6} + \frac{1}{4}$

Answer

$-\frac{7}{12}$

32) $-\frac{1}{7} + \frac{5}{8}$

33) $-\frac{8}{9} + \left(-\frac{1}{3}\right)$

Answer

$-\frac{11}{9}$

34) $-\frac{1}{3} + \left(-\frac{1}{2}\right)$

35) $-\frac{1}{4} - \left(-\frac{2}{9}\right)$

Answer

$-\frac{1}{36}$

36) $-\frac{1}{2}-\left(-\frac{1}{8}\right)$

37) $-\frac{8}{9}-\frac{4}{5}$

Answer

$-\frac{76}{45}$

38) $-\frac{4}{7}-\frac{1}{3}$

In Exercises 39-52, simplify the expression.

39) $\frac{8}{9}-\left|\frac{5}{2}-\frac{2}{5}\right|$

Answer

$-\frac{109}{90}$

40) $\frac{8}{5}-\left|\frac{7}{6}-\frac{1}{2}\right|$

41) $\left(-\frac{7}{6}\right)^2+\left(-\frac{1}{2}\right)\left(-\frac{5}{3}\right)$

Answer

$\frac{79}{36}$

42) $\left(\frac{3}{2}\right)^2+\left(-\frac{1}{2}\right)\left(\frac{5}{8}\right)$

43) $\left(-\frac{9}{5}\right)\left(-\frac{9}{7}\right)+\left(\frac{8}{5}\right)\left(-\frac{1}{2}\right)$

Answer

$\frac{53}{35}$

44) $\left(-\frac{1}{3}\right)\left(-\frac{5}{7}\right)+\left(\frac{2}{3}\right)\left(-\frac{6}{7}\right)$

45) $-\frac{5}{8}+\frac{7}{2}\left(-\frac{9}{2}\right)$

Answer

$-\frac{131}{8}$

46) $\frac{3}{2}+\frac{9}{2}\left(-\frac{1}{4}\right)$

47) $\left(-\frac{7}{5}\right)\left(\frac{9}{2}\right)-\left(-\frac{2}{5}\right)^2$

Answer

$-\frac{323}{50}$

48) $\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)-\left(\frac{1}{4}\right)^2$

49) $\frac{6}{5}-\frac{2}{5}\left(-\frac{4}{9}\right)$

Answer

$\frac{62}{45}$

50) $\frac{3}{2} - \frac{5}{6}\left(-\frac{1}{3}\right)$

51) $\left(\frac{2}{3}\right)\left(-\frac{8}{7}\right) - \left(\frac{4}{7}\right)\left(-\frac{9}{8}\right)$

Answer

$-\frac{5}{42}$

52) $\left(-\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{5}{8}\right)\left(-\frac{1}{8}\right)$

In Exercises 53-70, evaluate the expression at the given values.

53) $xy - z^2$ at $x = -1/2, y = -1/3$, and $z = 5/2$

Answer

$-\frac{73}{12}$

54) $xy - z^2$ at $x = -1/3, y = 5/6$, and $z = 1/3$

55) $-5x^2 + 2y^2$ at $x = 3/4$ and $y = -1/2$

Answer

$-\frac{37}{16}$

56) $-2x^2 + 4y^2$ at $x = 4/3$ and $y = -3/2$

57) $2x^2 - 2xy - 3y^2$ at $x = 3/2$ and $y = -3/4$

Answer

$\frac{81}{16}$

58) $5x^2 - 4xy - 3y^2$ at $x = 1/5$ and $y = -4/3$

59) $x + yz$ at $x = -1/3, y = 1/6$, and $z = 2/5$

Answer

$-\frac{2}{5}$

60) $x + yz$ at $x = 1/2, y = 7/4$, and $z = 2/3$

61) $ab + bc$ at $a = -4/7, b = 7/5$, and $c = -5/2$

Answer

$-\frac{43}{10}$

62) $ab + bc$ at $a = -8/5, b = 7/2$, and $c = -9/7$

63) x^3 at $x = -1/2$

Answer

$-\frac{1}{8}$

64) x^2 at $x = -3/2$

65) $x - yz$ at $x = -8/5, y = 1/3$, and $z = -8/5$

Answer

$-\frac{16}{15}$

66) $x - yz$ at $x = 2/3, y = 2/9$, and $z = -3/5$

67) $-x^2$ at $x = -8/3$

Answer

$-\frac{64}{9}$

68) $-x^4$ at $x = -9/7$

69) $x^2 + yz$ at $x = 7/2, y = -5/4$, and $z = -5/3$

Answer

$\frac{43}{3}$

70) $x^2 + yz$ at $x = 1/2, y = 7/8$, and $z = -5/9$

71) $a + b/c + d$ is equivalent to which of the following mathematical expressions?

a. $a + \frac{b}{c} + d$
b. $\frac{a+b}{c+d}$
c. $\frac{a+b}{c} + d$
d. $a + \frac{b}{c+d}$

Answer

(a)

72) $(a + b)/c + d$ is equivalent to which of the following mathematical expressions?

a. $a + \frac{b}{c} + d$
b. $\frac{a+b}{c+d}$
c. $\frac{a+b}{c} + d$
d. $a + \frac{b}{c+d}$

73) $a + b/(c + d)$ is equivalent to which of the following mathematical expressions?

a. $a + \frac{b}{c} + d$
b. $\frac{a+b}{c+d}$
c. $\frac{a+b}{c} + d$
d. $a + \frac{b}{c+d}$

Answer

(d)

74) (a + b)/(c + d) is equivalent to which of the following mathematical expressions?

a. $a + \frac{b}{c} + d$

b. $\dfrac{a+b}{c+d}$

c. $\dfrac{a+b}{c}+d$

d. $a+\dfrac{b}{c+d}$

75) Use the graphing calculator to reduce $4125/1155$ to lowest terms.

Answer

$\dfrac{25}{7}$

76) Use the graphing calculator to reduce $2100/945$ to lowest terms.

77) Use the graphing calculator to simplify: $\dfrac{45}{84}\cdot\dfrac{70}{33}$

Answer

$\dfrac{25}{22}$

78) Use the graphing calculator to simplify: $\dfrac{34}{55}+\dfrac{13}{77}$

79) Use the graphing calculator to simplify: $-\dfrac{28}{33}\div\left(-\dfrac{35}{44}\right)$

Answer

$\dfrac{16}{15}$

80) Use the graphing calculator to simplify: $-\dfrac{11}{84}-\left(-\dfrac{11}{36}\right)$

1.4 Decimal Notation

In Exercises 1-33, simplify the given expressions.

1) $-2.835+(-8.759)$

Answer

-11.594

2) $-5.2+(-2)$

3) $19.5-(-1.6)$

Answer

21.1

4) $9.174-(-7.7)$

5) $-2-0.49$

Answer

-2.49

6) $-50.86-9$

7) $(-1.2)(-0.05)$

Answer

0.06

8) $(-7.9)(0.9)$

9) $-0.13+23.49$

Answer

23.36

10) $-30.82+75.93$

11) $16.4+(-41.205)$

Answer

-24.805

12) $-7.8+3.5$

13) $-0.4508 \div 0.49$

Answer

-0.92

14) $0.2378 \div(-0.29)$

15) $(-1.42)(-3.6)$

Answer

5.112

16) $(-8.64)(4.6)$

17) $2.184 \div(-0.24)$

Answer

-9.1

18) $7.395 \div(-0.87)$

19) $(-7.1)(-4.9)$

Answer

34.79

20) $(5.8)(-1.9)$

21) $7.41 \div(-9.5)$

Answer

-0.78

22) $-1.911 \div 4.9$

23) $-24.08 \div 2.8$

Answer

-8.6

24) $61.42 \div(-8.3)$

25) $(-4.04)(-0.6)$

Answer

2.424

26) $(-5.43)(0.09)$

27) $-7.2-(-7)$

Answer

-0.2

28) $-2.761-(-1.5)$

29) $(46.9)(-0.1)$

Answer

-4.69

30) $(-98.9)(-0.01)$

31) $(86.6)(-1.9)$

Answer

-164.54

32) $(-20.5)(8.1)$

In Exercises 33-60, simplify the given expression.

33) $-4.3-(-6.1)(-2.74)$

Answer

-21.014

34) $-1.4-1.9(3.36)$

35) $-3.49+|-6.9-(-15.7)|$

Answer

5.31

36) $1.3+|-13.22-8.79|$

37) $|18.9-1.55|-|-16.1-(-17.04)|$

Answer

16.41

38) $|-17.5-16.4|-|-15.58-(-4.5)|$

39) $8.2-(-3.1)^3$

Answer

37.991

40) $-8.4-(-6.8)^3$

41) $5.7-(-8.6)(1.1)^2$

Answer

16.106

42) $4.8-6.3(6.4)^2$

43) $(5.67)(6.8)-(1.8)^2$

Answer

35.316

44) $(-8.7)(8.3)-(-1.7)^2$

45) $9.6+(-10.05-13.16)$

Answer

-13.61

46) $-4.2+(17.1-14.46)$

47) $8.1+3.7(5.77)$

Answer

29.449

48) $8.1+2.3(-5.53)$

49) $7.5+34.5/(-1.6+8.5)$

Answer

12.5

50) $-8.8+0.3/(-7.2+7.3)$

51) $(8.0+2.2)/5.1-4.6$

Answer

\(− 2.6\)

52) $(35.3+1.8)/5.3-5.4$

53) $-18.24-|-18.5-19.7|$

Answer

-56.44

54) $16.8-|4.58-17.14|$

55) $-4.37-|-8.97|$

Answer

-13.34

56) $4.1-|-8.4|$

57) $7.06-(-1.1-4.41)$

Answer

12.57

58) $7.74-(0.9-7.37)$

59) $-2.2-(-4.5)^2$

Answer

-22.45

60) $-2.8-(-4.3)^2$

61) Evaluate $a-b^2$ at $a=-2.9$ and $b=-5.4$.

Answer

-32.06

62) Evaluate $a-b^3$ at $a=-8.3$ and $b=-6.9$.

63) Evaluate $a+|b-c|$ at $a=-19.55$, $b=5.62$, and $c=-5.21$.

Answer

-8.72

64) Evaluate $a-|b-c|$ at $a=-8.37$, $b=-8.31$, and $c=17.5$.

65) Evaluate $a-bc$ at $a=4.3$, $b=8.5$, and $c=1.73$.

Answer

-10.405

66) Evaluate $a+bc$ at $a=4.1$, $b=3.1$, and $c=-7.03$.

67) Evaluate $a-(b-c)$ at $a=-7.36$, $b=-17.6$, and $c=-19.07$.

Answer

-8.83

68) Evaluate $|a-b|-|c-d|$ at $a=1.91$, $b=19.41$, $c=-11.13$, and $d=4.3$.

69) Evaluate $a+b/(c+d)$ at $a=4.7$, $b=54.4$, $c=1.7$, and $d=5.1$.

Answer

12.7

70) Evaluate $(a+b)/c-d$ at $a=-74.2$, $b=3.8$, $c=8.8$, and $d=7.5$.

71) Evaluate $ab-c^2$ at $a=-2.45$, $b=5.6$, and $c=-3.2$.

Answer

-23.96

72) Evaluate $a+(b-c)$ at $a=12.6$, $b=-13.42$, and $c=-15.09$.

73) Evaluate $a-|b|$ at $a=-4.9$ and $b=-2.67$.

Answer

-7.57

74) Evaluate $a-bc^2$ at $a=-3.32$, $b=-5.4$, and $c=-8.5$.

75) Use your graphing calculator to evaluate $3.5-1.7x$ at $x=1.25$. Round your answer to the nearest tenth.

Answer

1.4

76) Use your graphing calculator to evaluate $2.35x - 1.7$ at $x = -12.23$. Round your answer to the nearest tenth.

77) Use your graphing calculator to evaluate $1.7x^2 - 3.2x + 4.5$ at $x = 2.86$. Round your answer to the nearest hundredth.

Answer

9.25

78) Use your graphing calculator to evaluate $19.5 - 4.4x - 1.2x^2$ at $x = -1.23$. Round your answer to the nearest hundredth.

79) Use your graphing calculator to evaluate $-18.6 + 4.4x^2 - 3.2x^3$ at $x = 1.27$. Round your answer to the nearest thousandth.

Answer

-4.948

80) Use your graphing calculator to evaluate $-4.4x^3 - 7.2x - 18.2$ at $x = 2.29$. Round your answer to the nearest thousandth.

1.5 Algebraic Expressions

In Exercises 1-6, use the associative property of multiplication to simplify the expression.

Note: ***You must show the regrouping step using the associative property on your homework.***

1) $-3(6a)$

Answer

$-18a$

2) $-10(2y)$

3) $-9(6ab)$

Answer

$-54ab$

4) $8(5xy)$

5) $-7\left(3x^2\right)$

Answer

$-21x^2$

6) $-6(8z)$

In Exercises 7-18, use the distributive property to expand the given expression.

7) $4(3x - 7y)$

Answer

$12x - 28y$

8) $-4(5a + 2b)$

9) $-6(-y + 9)$

Answer

$6y - 54$

10) $5(-9w + 6)$

11) $-9(s + 9)$

Answer

$-9s-81$

12) $6(-10y+3)$

13) $-(-3u-6v+8)$

Answer

$3u+6v-8$

14) $-(3u-3v-9)$

15) $-8\left(4u^2-6v^2\right)$

Answer

$-32u^2+48v^2$

16) $-5(8x-9y)$

17) $-(7u+10v+8)$

Answer

$-7u-10v-8$

18) $-(7u-8v-5)$

In Exercises 19-26, combine like terms by first using the distributive property to factor out the common variable part, and then simplifying.

Note: ***You must show the factoring step on your homework.***

19) $-19x+17x-17x$

Answer

$-19x$

20) $11n-3n-18n$

21) $14x^3-10x^3$

Answer

$4x^3$

22) $-11y^3-6y^3$

23) $9y^2x+13y^2x-3y^2x$

Answer

$19y^2x$

24) $4x^3-8x^3+16x^3$

25) $15m+14m$

Answer

$29m$

26) $19q+5q$

In Exercises 27-38, simplify each of the following expressions by rearranging and combining like terms mentally.

Note: ***This means write down the problem, then write down the answer. No work.***

27) $9-17m-m+7$

Answer

$16-18m$

28) $-11+20x+16x-14$

29) $-6y^2-3x^3+4y^2+3x^3$

Answer

$-2y^2$

30) $14y^3-11y^2x+11y^3+10y^2x$

31) $-5m-16+5-20m$

Answer

$-25m-11$

32) $-18q+12-8-19q$

33) $-16x^2y+7y^3-12y^3-12x^2y$

Answer

$-28x^2y-5y^3$

34) $10x^3+4y^3-13y^3-14x^3$

35) $-14r+16-7r-17$

Answer

$-21r-1$

36) $-9s-5-10s+15$

37) $14-16y-10-13y$

Answer

$4-29y$

38) $18+10x+3-18x$

In Exercises 39-58, use the distributive property to expand the expression, then combine like terms mentally.

39) $3-(-5y+1)$

Answer

$2+5y$

40) $5-(-10q+3)$

41) $-\left(9y^2+2x^2\right)-8\left(5y^2-6x^2\right)$

Answer

$-49y^2+46x^2$

42) $-8\left(-8y^2+4x^3\right)-7\left(3y^2+x^3\right)$

43) $2(10-6p)+10(-2p+5)$

Answer

$70-32p$

44) $2(3-7x)+(-7x+9)$

45) $4(-10n+5)-7(7n-9)$

Answer

$-89n+83$

46) $3(-9n+10)+6(-7n+8)$

47) $-4x-4-(10x-5)$

Answer

$-14x+1$

48) $8y+9-(-8y+8)$

49) $-7-(5+3x)$

Answer

$-12-3x$

50) $10-(6-4m)$

51) $-8(-5y-8)-7(-2+9y)$

Answer

$-23y+78$

52) $6(-3s+7)-(4-2s)$

53) $4\left(-7y^2-9x^2y\right)-6\left(-5x^2y-5y^2\right)$

Answer

$2y^2-6x^2y$

54) $-6\left(x^3+3y^2x\right)+8\left(-y^2x-9x^3\right)$

55) $6s-7-(2-4s)$

Answer

$10s-9$

56) $4x-9-(-6+5x)$

57) $9(9-10r)+(-8-2r)$

Answer

$73-92r$

58) $-7(6+2p)+5(5-5p)$

In Exercises 59-64, use the distributive property to simplify the given expression.

59) $-7x+7(2x-5[8x+5])$

Answer

$-273x-175$

60) $-9x+2(5x+6[-8x-3])$

61) $6x-4(-3x+2[5x-7])$

Answer

$-22x+56$

62) $2x+4(5x-7[8x+9])$

63) $-8x-5(2x-3[-4x+9])$

Answer

$-78x+135$

64) $8x+6(3x+7[-9x+5])$

Contributors and Attributions

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CHAPTER OVERVIEW

2: Solving Linear Equations and Inequalities

2.1: Solving Equations - One Step
2.2: Solving Equations - Multiple Steps
2.3: Clearing Fractions and Decimals
2.4: Formulae
2.5: Applications
2.6: Inequalities
2.E: Solving Linear Equations and Inequalities (Exercises)

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2.1: Solving Equations - One Step

Let's start with the definition of an equation.

Equation

An equation is a mathematical statement that equates two algebraic expressions.

The key difference between a algebraic expression and an equation is the presence of an an equals sign. So, for example,

$$2x+3, \quad x-(3-2x), \quad \text{and} \quad 2(y+3)-3(1-y)$$

are algebraic expressions, while

$$2x+3=0, \quad x-(3-2x)=4, \quad \text{and} \quad 2(y+3)-3(1-y)=-11$$

are equations. Note that each of the equations contain an equals sign, but the algebraic expressions do not.

Next we have the definition of a solution of an equation.

What it Means to be a Solution of an Equation

A solution of an equation is a numerical value that satisfies the equation. That is, when the variable in the equation is replaced by the solution, a true statement results.

Example $2.1.1$

Show that 8 is a solution of the equation $x-12=-4$.

Solution

Substitute 8 for x in the given equation and simplify.

$$\begin{aligned} x-12&=-4 && \text{The given equation.} \\ 8-12&=-4 && \text{Substitute 8 for } x. \\ -4&=-4 && \text{Simplify both sides.} \end{aligned}$$

Since the left- and right-hand sides of the last line are equal, this shows that when 8 is substituted for x in the equation a true statement results. Therefore, 8 is a solution of the equation

Exercise $2.1.1$

Which of the numbers $\{1,2,3,4,5\}$ is a solution of the equation $2y+3=7$.

Answer

2

Equivalent Equations

Now that we know how to identify a solution of an equation, let's define what it means when we say that two equations are equivalent.

Equivalent Equations

Two equations are equivalent if they have the same solution set.

Example 2.1.2

Are the equations $x-3=6$ and $x=9$ equivalent?

Solution

The number 9 is the only solution of the equation $x-3=6$. Similarly, 9 is the only solution of the equation $x=9$. Therefore $x-3=6$ and $x=9$ have the same solution sets and are equivalent.

Exercise 2.1.2

Are the equations $x=5$ and $x-7=10$ equivalent?

Answer

No

Example 2.1.3

Are the equations $x^2=1$ and $x=1$ equivalent?

Solution

By inspection, the equation $x^2=1$ has two solutions, -1 and 1.

$$(-1)^2=1 \quad \text{and} \quad (1)^2=1$$

On the other hand, the equation $x=1$ has a single solution, namely 1. Hence, the equations $x^2=1$ and $x=1$ do not have the same solution sets and are not equivalent.

Exercise 2.1.3

Are the equations $x=1$ and $x^2=x$ equivalent?

Answer

No

As we shall soon see, equivalent equations play an important role in finding the solution of an equation.

Wrap and Unwrap, Do and Undo

Suppose that you are wrapping a gift for your cousin. You perform the following steps in order.

1. Put the gift paper on.
2. Put the tape on.
3. Put the decorative bow on.

When we give the wrapped gift to our cousin, he politely unwraps the present, “undoing” each of our three steps in inverse order.

1. Take off the decorative bow.
2. Take off the tape.
3. Take off the gift paper.

This seemingly frivolous wrapping and unwrapping of a gift contains some deeply powerful mathematical ideas. Consider the mathematical expression $x+4$. To evaluate this expression at a particular value of x, we would start with the given value of x, then add 4.

- Let’s set x equal to the number 7. If we add 4, we arrive at the following result: 11

Now, how would we “unwrap” this result to return to our original number? We would start with our result, namely 11, then subtract 4.

- Take our result from above, 11. If we subtract 4, we return to our original value of x: 7

The above discussion leads us to two extremely important observations.

Note

The inverse of addition is subtraction. If we start with a number x and add a number a, then subtracting a from the result will return us to the original number x. In symbols,

$$x+a-a=x$$

That is, subtracting a “undoes” the effect of adding a and returns us to the original number x.
The inverse of subtraction is addition. If we start with a number x and subtract a number a, then adding a to the result will return us to the original number x. In symbols,

$$x-a+a=x$$

That is, adding a “undoes” the effect of subtracting a and returns us to the original number x.

Operations that Produce Equivalent Equations

In Example 2.1.1, we saw that $x=8$ was a solution of the equation $x-12=-4$. Indeed, the equations $x=8$ and $x-12=-4$ are equivalent equations because both have the same solution sets.

In the case of the equation $x-12=-4$, it’s fairly easy to “guess” that the solution is $x=8$, but as the equations become more complicated, we will want to know just what operations we can perform on the equation that will not change the solution set. The goal is to start with an equation such as

$$x-12=-4$$

then through a series of steps that do not change the solution, arrive at the solution

$$x=8$$

With these thoughts in mind, there are a number of operations that will produce equivalent equations (equations with the same solution set). The first two that we will employ are adding or subtracting the same amount from both sides of an equation.

Adding the Same Quantity to Both Sides of an Equation

Adding the same quantity to both sides of an equation does not change the solution set. That is, if

$$a=b$$

then adding c to both sides of the equation produces the equivalent equation

$$a+c=b+c$$

Subtracting the Same Quantity from Both Sides of an Equation

Subtracting the same quantity to both sides of an equation does not change the solution set. That is, if

$$a=b$$

then subtracting c from both sides of the equation produces the equivalent equation

$$a-c=b-c$$

Let’s look at an example where adding the same amount to both sides of the equation produces an equivalent equation that is the solution to the original equation.

Example 2.1.4

Solve $x-7=12$ for x.

Solution

To undo the effect of subtracting 7, we add 7 to both sides of the equation.

$$\begin{aligned} x-7&=12 && \text{Original equation.} \\ x-7+7&=12+7 && \text{Adding 7 to both sides of the equation produces an equivalent equation.} \end{aligned}$$

$$x = 19 \quad \text{On the left, adding 7 "undoes" the effect of subtracting 7 and returns } x\text{. On the right, } 12 + 7 = 19$$

Therefore, the solution of the equation is 19.

Check: To check, substitute the solution 19 into the original equation.

$$\begin{aligned} x - 7 &= 12 && \text{Original equation.} \\ 19 - 7 &= 12 && \text{Substitute 19 for } x \\ 12 &= 12 && \text{Simplify both sides.} \end{aligned}$$

The fact that the last line of the check is a true statement guarantees that 19 is a solution of $x - 7 = 12$.

Exercise 2.1.4

Solve for $x : x - 6 = 4$.

Answer

10

In the solution of Example $2.1.4$, we use the concept of the "inverse." If we start with x, subtract 7, then add 7, we are returned to the number x. In symbols, $x - 7 + 7 = x$. We are returned to x because "subtracting 7" and "adding 7" are inverse operations of one another. That is, whatever one does, the other "undoes."

Next, let's look at an example where subtracting the same amount from both sides of the equation produces an equivalent equation that is the solution to the original equation.

Example 2.1.5

Solve $x + \frac{2}{3} = \frac{1}{2}$ for x.

Solution

To undo the effect of adding $2/3$, we subtract $2/3$ from both sides of the equation.

$$\begin{aligned} x + \frac{2}{3} &= \frac{1}{2} && \text{Original equation.} \\ x + \frac{2}{3} - \frac{2}{3} &= \frac{1}{2} - \frac{2}{3} && \text{Subtracting } 2/3 \text{ from both sides produces an equivalent equation.} \\ x &= \frac{3}{6} - \frac{4}{6} && \text{On the left, subtracting } 2/3 \text{ "undoes" the effect of adding } 2/3 \text{ and returns } x. \\ & && \text{On the right, make equivalent fractions with a common denominator.} \\ x &= -\frac{1}{6} && \text{Subtract: } \frac{3}{6} - \frac{4}{6} = -\frac{1}{6} \end{aligned}$$

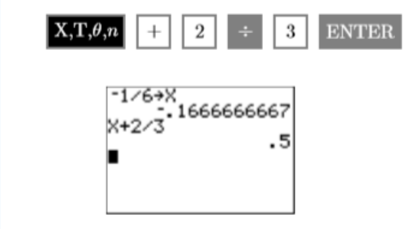
Therefore, the solution of the equation is $-1/6$.

Check: Let's use the TI-84 graphing calculator to check this solution.

1. Store the value $-1/6$ in the variable X using the following keystrokes. The result is shown in Figure $2.1.1$.

(-) 1 ÷ 6 STO▷ X,T,θ,n ENTER

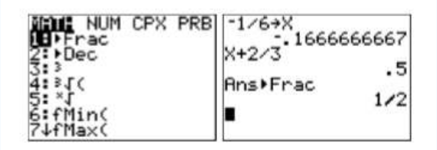
2. Enter the left-hand side of the original equation: x +2 /3. Use the following keystrokes. The result is shown in Figure 2.1.



```
-1/6→X
          -.1666666667
X+2/3
                    .5
```

Figure 2.1.1: Checking the solution to $x+2/3=1/2$.

3. Press the MATH button on your calculator (see Figure 2.1.2), then select 1:Frac and press the ENTER button. This will convert the decimal result to a fraction (see Figure 2.1.2).



```
-1/6→X
          -.1666666667
X+2/3
                    .5
Ans▸Frac
                   1/2
```

Figure 2.1.2: Changing the result to a fraction.

Note that the result is $\frac{1}{2}$, showing that $-\frac{1}{6}$ is a solution of $x+\frac{2}{3}=\frac{1}{2}$.

Exercise 2.1.5

Solve for $x: x+\frac{1}{2}=\frac{3}{5}$.

Answer

$1/10$

More Operations That Produce Equivalent Equations

Here are two more operations that produce equivalent equations.

Multiplying Both Sides of an Equation by a Nonzero Quantity

Multiplying both sides of an equation by a nonzero quantity does not change the solution set. That is, if

$$a=b$$

and $c \neq 0$, then multiplying both sides of the equation by c produces the equivalent equation

$$ac=bc$$

Dividing Both Sides of an Equation by a Nonzero Quantity

Dividing both sides of an equation by a nonzero quantity does not change the solution set. That is, if

$$a=b$$

and $c \neq 0$, then dividing both sides of the equation by c produces the equivalent equation

$$\frac{a}{c}=\frac{b}{c}$$

Like addition and subtraction, multiplication and division are inverse operations.

Note

The inverse of multiplication is division. If we start with a number x and multiply by a number a, then dividing the result by the same number a returns us to the original number x. In symbols,

$$\frac{a \cdot x}{a}=x$$

That is, dividing by a "undoes" the effect of multiplying by a and returns us to the original number x.

The inverse of division is multiplication. If we start with a number x and divide by a number a, then multiplying the result by the same number a returns us to the original number x. In symbols,

$$a \cdot \frac{x}{a} = x$$

That is, multiplying by a “undoes” the effect of dividing by a and returns us to the original number x.

Let’s look at an example where dividing both sides of the equation by the same amount produces an equivalent equation that is the solution to the original equation.

Example 2.1.6

Solve $-2.1x = 0.42$ for x.

Solution

To undo the effect of multiplying by -2.1, we divide both sides of the equation by -2.1.

$$\begin{aligned} &-2.1 \\ &\quad x = 0.42 \quad \color{red}{\text{Original equation.}} \\ &-2.1 \\ &\frac{x}{-2.1} = \frac{0.42}{-2.1} \quad \color{red}{\text{Dividing both sides by } -2.1 \text{ produces an equivalent equation.}} \\ &\quad x = -2 \quad \color{red}{\text{On the left, dividing by } -2.1\text{"undoes" the effect of multiplying by } -2.1 \text{ and}} \\ &\quad \color{red}{\text{returns } x.\text{On the right, divide: } 0.42/(-2.1) = -2} \end{aligned}$$

Therefore, the solution of the equation is -2.

Check: To check, substitute the solution -2 into the original equation.

$$\begin{aligned} -2.1x &= 0.42 \quad && \color{red}{\text{Original equation.}} \\ -2.1(-2) &= 0.42 \quad && \color{red}{\text{Substitute } -2 \text{ for } x} \\ 0.42 &= 0.42 \quad && \color{red}{\text{On the left, multiply: } -2.1(-2) = 0.42} \end{aligned}$$

The fact that the last line of the check is a true statement guarantees that -2 is a solution of $-2.1x = 0.42$.

Exercise 2.1.6

Solve for $x : -3.6x = 0.072$.

Answer

-0.02

Next, lets look at an example where multiplying both sides of the equation by the same amount produces an equivalent equation that is the solution to the original equation.

Example 2.1.7

Solve $\frac{x}{5} = -10$ for x.

Solution

To undo the effect of dividing by 5, we multiply both sides of the equation by 5.

$$\begin{aligned} \frac{x}{5} &= -10 \quad \color{red}{\text{Original equation.}} \\ 5\left[\frac{x}{5}\right] &= [-10]5 \quad \color{red}{\text{Multiplying both sides by 5produces an equivalent equation.}} \\ x &= -50 \quad \color{red}{\text{On the left, multiplying by 5 "undoes" the effect of dividing by 5 and}} \\ & \quad \color{red}{\text{returns } x.\text{ On the right, multiply: } [-10]5 = -50} \end{aligned}$$

Therefore, the solution of the equation is -50.

Check: Let’s use the TI-84 graphing calculator to check this solution.

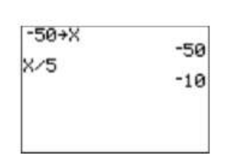
1. Store the value -50 in the variable X using the following keystrokes. The result is shown in Figure 2.1.3.

(-) 5 0 STO▷ X,T,θ,n ENTER

2. Enter the left-hand side of the original equation: $\frac{x}{5}$. Use the following keystrokes.

X,T,θ,n ÷ 5 ENTER

Figure 2.1.3.



```
-50→X
                -50
X/5
                -10
```

Figure 2.1.3: Checking the solution to $x/5=-10$.

Note that the result is -10, showing that -50 is a solution of $x/5=-10$.

Exercise 2.1.7

Solve for x : $\frac{x}{7}=-2$.

Answer

-14

Writing Mathematics

When solving equations, observe the following rules when presenting your work.

1. **One equation per line**. This means that you should not arrange your work horizontally as follows:

$$x+3=7 \quad x+3-3=7-3 \quad x=4$$

That's three equations on a line. Instead, work vertically, writing one equation per line. In the following presentation, note how we align the equal signs in a column.

$$\begin{aligned} x+3&=7\\ x+3-3&=7-3\\ x&=4 \end{aligned}$$

2. **Add and subtract inline**. Do not add 7 to both sides of the equation in the following manner:

$$\begin{array}{r} x-7=12\\ +7 \quad +7\\ \hline x=19 \end{array}$$

Instead, add 7 "inline" to both sides of the equation as follows:

$$\begin{aligned} x-7&=12\\ x-7+7&=12+7\\ x&=19 \end{aligned}$$

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2.2: Solving Equations - Multiple Steps

Recall the "Wrap" and "Unwrap" discussion from Section 2.1. To wrap a present we put the gift paper on, put the tape on, and put the decorative bow on. To unwrap the gift, we must "undo" each of these steps in inverse order. Hence, to unwrap the gift we take off the decorative bow, take off the tape, and take off the gift paper.

Now, imagine a machine that multiplies its input by 3, then adds 5 to the result. This machine is pictured on the left in Figure $2.2.1$.

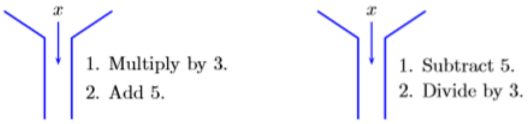


***Figure** $2.2.1$**:** The second machine "unwraps" the first machine.*

To "unwrap" the effect of the machine on the left, we need a machine that will "undo" each of the steps of the first machine, but in inverse order. The "unwrapping" machine is pictured on the right in Figure $2.2.1$. It will first subtract 5 from its input, then divide the result by 3. Note that each of these operations "undoes" the corresponding operation of the first machine, but in inverse order.

The following argument shows that the second machine "undoes" the operation of the first machine.

1. Drop the integer 4 into the machine on the left in Figure $2.2.1$. This machine will first multiply 4 by 3, then add 5 to the result. The result is $3(4)+5$, or 17.
2. To "unwrap" this result, drop 17 into the machine on the right. This machine first subtracts 5, then divides the result by 3. The result is $(17-5)/3$, or 4, the original integer that was put into the first machine.

Example $2.2.1$

Solve for $x: 3x+5=14$.

Solution

On the left, order of operations demands that we first multiply x by 3, then add 5. To solve this equation for x, we must "undo" each of these operations in inverse order. Thus, we will first subtract 5 from both sides of the equation, then divide both sides by 3.

$$\begin{aligned} 3x+5&=14 && \text{Original equation.} \\ 3x+5-5&=14-5 && \text{To "undo" adding 5, subtract 5 from both sides of the equation.} \\ 3x&=9 && \text{Simplify both sides.} \\ \frac{3x}{3}&=\frac{9}{3} && \text{To "undo" multiplying by 3, divide both sides of the equation by 3.} \\ x&=3 && \text{Simplify both sides.} \end{aligned}$$

Check: To check the solution, substitute 3 for x in the original equation and simplify.

$$\begin{aligned} 3x+5&=14 && \text{Original equation.} \\ 3(3)+5&=14 && \text{Substitute 3 for x.} \\ 9+5&=14 && \text{Multiply first: } 3(3)=9. \\ 14&=14 && \text{Add: } 9+5=14. \end{aligned}$$

Because the last line of the check is a true statement, this guarantees that 3 is a solution of the original equation.

Exercise $2.2.1$

Solve for $x: 2x+3=7$.

Answer

$x=2$

Let's try an equation with fractions.

Example $2.2.2$

Solve for $x: \frac{x}{5} - \frac{2}{3} = \frac{1}{2}$.

Solution

On the left, order of operations demands that we first divide x by 5, then subtract $2/3$. To solve this equation for x, we must “undo” each of these operations in inverse order. Thus, we will first add $2/3$ to both sides of the equation, then multiply both sides of the resulting equation by 5.

$$\begin{aligned} \frac{x}{5} - \frac{2}{3} &= \frac{1}{2} && \text{Original equation.} \\ \frac{x}{5} - \frac{2}{3} + \frac{2}{3} &= \frac{1}{2} + \frac{2}{3} && \text{To "undo" subtracting } 2/3\text{, add } 2/3 \text{ to both sides of the equation.} \end{aligned}$$

On the left, we simplify. On the right, we make equivalent fractions with a common denominator.

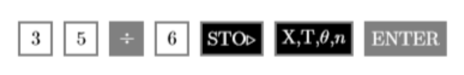
$$\begin{aligned} \frac{x}{5} &= \frac{3}{6} + \frac{4}{6} && \text{Make equivalent fractions} \\ \frac{x}{5} &= \frac{7}{6} && \text{Add: } \frac{3}{6} + \frac{4}{6} = \frac{7}{6} \end{aligned}$$

Now we “undo” dividing by five by multiplying both sides of the equation by 5.

$$\begin{aligned} \left(\frac{x}{5}\right) &= 5\left(\frac{7}{6}\right) && \text{Multiply both sides by 5} \\ x &= \frac{35}{6} && \text{On the left, simplify. On the right, multiply: } 5\left(\frac{7}{6}\right) = \frac{35}{6} \end{aligned}$$

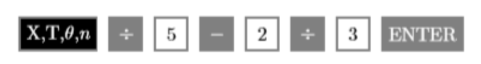
Check: Let’s use the TI-84 to check this solution.

1. Store the value $35/6$ in the variable X using the following keystrokes.

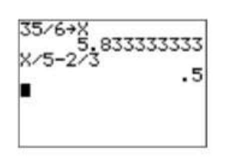


   The result is shown in Figure $2.2.2$.

2. Enter the left-hand side of the original equation: $x/5 - 2/3$. Use the following keystrokes.



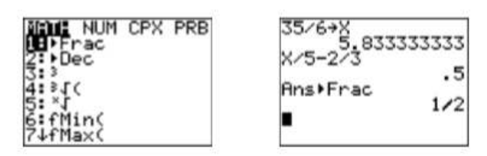
   The result is shown in Figure $2.2.2$.



```
35/6→X
         5.833333333
X/5-2/3
                  .5
■
```

***Figure** 2.2.2**: Checking the solution to x/5−2/3=1/2.*

3. Press the MATH button on your calculator (see Figure $2.2.3$), then select 1:►Frac, then press the ENTER button. This will convert the decimal result to a fraction (see Figure $2.2.3$).



```
35/6→X
         5.833333333
X/5-2/3
                  .5
Ans▸Frac
                 1/2
■
```

***Figure** 2.2.3**: Changing the result to a fraction.*

Note that the result is $\frac{1}{2}$, showing that $\frac{35}{6}$ is a solution of $\frac{x}{5} - \frac{2}{3} = \frac{1}{2}$.

Exercise 2.2.2

Solve for $x: \frac{x}{2}-\frac{3}{5}=\frac{1}{4}$.

Answer

$x=17/10$

Let’s try an equation with decimals.

Example 2.2.3

Solve for $x: 5.2x+2.3=-3.94$.

Solution

On the left, order of operations demands that we first multiply x by 5.2, then add 2.3. To solve this equation for x, we must “undo” each of these operations in inverse order. Thus, we will first subtract 2.3 from both sides of the equation, then divide both sides by 5.2.

$$\begin{aligned} 5.2x+2.3 &= -3.94 && \text{Original equation.} \\ 5.2x+2.3-2.3 &= -3.94-2.3 && \text{To "undo" adding 2.3, subtract 2.3 from both sides.} \\ 5.2x &= -6.24 && \text{On the left, simplify. On the right, add: -3.94-2.3=-6.24.} \\ \frac{5.2x}{5.2} &= \frac{-6.24}{5.2} && \text{To undo multiplying by 5.2, divide both sides by 5.2.} \\ x &= -1.2 && \text{On the left, simplify. On the right, divide: -6.24/5.2=-1.2.} \end{aligned}$$

Check: To check the solution, substitute -1.2 for x in the original equation and simplify.

$$\begin{aligned} 5.2x+2.3 &= -3.94 && \text{Original equation.} \\ 5.2(-1.2)+2.3 &= -3.94 && \text{Substitute -1.2 for x.} \\ -6.24+2.3 &= -3.94 && \text{Multiply: 5.2(-1.2)=-6.24} \\ -3.94 &= -3.94 && \text{Add: -6.24+2.3=-3.94} \end{aligned}$$

Because the last line of the check is a true statement, this guarantees that -1.2 is a solution of the original equation.

Exercise 2.2.3

Solve for $x: 3.25-1.2x=0.37$.

Answer

$x=2.4$

Variables on Both Sides of the Equation

It is not uncommon that the variable you are solving for appears in terms on both sides of the equation. Consider, for example, the equation $2x+3=5-7x$. In cases like this, it is helpful to have a general understanding of what it means to “solve for x.”

Solve for x

When asked to solve an equation for x, the goal is to manipulate the equation into the final form

$$x=\text{"Stuff"}$$

where “Stuff” is a valid mathematical expression that may contain other variables, mathematical symbols, etc., but it must not contain any occurrence of the variable x.

In this section, “Stuff” will always be a single number, but in Section 2.4, Formulae, “Stuff” will take on added complexity, including variables other than x.

Strategy for solving for x

When asked to solve an equation for x, a common strategy is to isolate all terms containing the variable x on one side of the equation and move all terms not containing the variable x to the other side of the equation.

Example 2.2.4

Solve $3-2x=5x+9$ for x.

Solution

We need to isolate all terms containing x on one side of the equation. We can eliminate $5x$ from the right-hand side of $3-2x=5x+9$ by subtracting $5x$ from both sides of the equation.

$$\begin{aligned} 3-2x &= 5x+9 \quad \text{Original equation.} \\ 3-2x-5x &= 5x+9-5x \quad \text{Subtract } 5x \text{ from both sides.} \\ 3-7x &= 9 \quad \text{Simplify both sides.} \end{aligned}$$

Next, eliminate 3 from the left-hand side of the last equation by subtracting 3 from both sides of the equation.

$$\begin{aligned} 3-7x-3 &= 9-3 \quad \text{Subtract 3 from both sides.} \\ -7x &= 6 \quad \text{Simplify both sides.} \end{aligned}$$

Note how we have isolated all terms containing x on one side of the equation.

$$\begin{aligned} \frac{-7x}{-7} &= \frac{6}{-7} \quad \text{Divide both sides by } -7 \\ x &= -\frac{6}{7} \quad \text{Simplify both sides.} \end{aligned}$$

Check: To check the solution, substitute $-6/7$ for x in the original equation.

$$\begin{aligned} 3-2x &= 5x+9 \quad \text{Original equation.} \\ 3-2\left(-\frac{6}{7}\right) &= 5\left(-\frac{6}{7}\right)+9 \quad \text{Substitute -6/7 for x.} \\ 3+\frac{12}{7} &= -\frac{30}{7}+9 \quad \text{Multiply: } -2(-6/7)=12/7 \text{ and } 5(-6/7)=-30/7. \\ \frac{21}{7}+\frac{12}{7} &= -\frac{30}{7}+\frac{63}{7} \quad \text{Make equivalent fractions with a common denominator.} \\ \frac{33}{7} &= \frac{33}{7} \quad \text{Add.} \end{aligned}$$

Because the last line of the check is a true statement, this guarantees that $-6/7$ is a solution of the original equation.

Exercise 2.2.4

Solve for $x: 4x+7=5-8x$.

Answer

$x=-1/6$

Simplifying Expressions when Solving Equations

Sometimes we need to simplify expressions before we can isolate terms containing x.

Example 2.2.5

Solve for $x: 2(3x+1)-3(4-2x)=-34$.

Solution

We'll first simplify the expression on the left-hand side of the equation using the *Rules Guiding Order of Operations*.

$$\begin{aligned} 2(3x+1)-3(4-2x) &= -34 \quad \text{Original equation.} \\ 6x+2-12+6x &= -34 \\ &\quad \text{Multiply: } 2(3x+1)=6x+2, \text{ Multiply: } -3(4-2x)=-12+6x \\ 12x-10 &= -34 \quad \text{Add: } 6x+6x=12x, \text{ Add: } 2-12=-10 \end{aligned}$$

To "undo" subtracting 10, we add 10 to both sides of the equation.

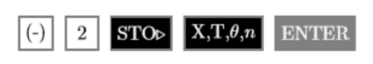
$$\begin{aligned} 12x-10+10 &= -34+10 \quad \text{Add 10 to both sides.} \\ 12x &= -24 \quad \text{Simplify both sides.} \end{aligned}$$

To "undo" multiplying by 12, we divide both sides by 12.

$$\begin{aligned} \frac{12x}{12} &= \frac{-24}{12} && \color{red}{\text{Divide both sides by 12}} \\ x &= -2 && \color{red}{\text{Simplify both sides.}} \end{aligned}$$

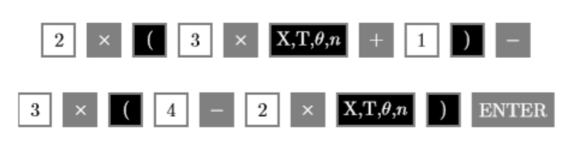
Check: Let's use the TI-84 to check this solution.

1. First, store −2 in the variableX using the following keystrokes.

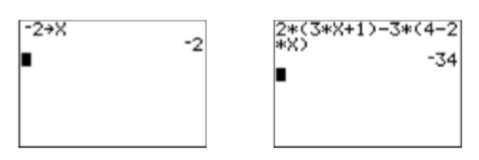


The result is shown in the first image in Figure 2.2.4.

2. Enter the left-hand side of original equation: $2(3x+1)-3(4-2x)$. Use the following keystrokes.



The result is shown in the second image in Figure 2.2.4.



```
-2→X
                -2
```

```
2*(3*X+1)-3*(4-2
*X)
               -34
```

***Figure** 2.2.4**:** Checking the solution to* $2(3x+1)-3(4-2x)=-34$.

Note that when -2 is substituted for x in the left-hand of the equation, the result is -34, equalling the right-hand side of the equation. Thus, the solution -2 checks.

Exercise 2.2.5

Solve for $x: 2x-(x-2)=2(x+7)$.

Answer

$x=-12$

Example 2.2.6

Solve for $x: 2x-5(3-2x)=4(x-1)$.

Solution

We'll first simplify the expressions on each side of the equation using the *Rules Guiding Order of Operations*.

$$\begin{aligned} 2x-5(3-2x) &= 4(x-1) && \color{red}{\text{Original equation.}} \\ 2x-15+10x &= 4x-4 && \color{red}{\text{On the left, distribute the -5. On the right, distribute the 4.}} \\ 12x-15 &= 4x-4 && \color{red}{\text{On the left, add: } 2x+10x=12x} \end{aligned}$$

Next, we need to isolate terms containing the variable x on one side of the equation. To remove the term $4x$ from the right-hand side, we subtract $4x$ from both sides of the equation.

$$\begin{aligned} 12x-15-4x &= 4x-4-4x && \color{red}{\text{Subtract } 4x \text{ from both sides.}} \\ 8x-15 &= -4 && \color{red}{\text{Simplify both sides.}} \end{aligned}$$

To remove the term -15 from the left-hand side, we add 15 to both sides of the equation.

$$\begin{aligned} 8x-15+15 &= -4+15 && \color{red}{\text{Add 15 to both sides.}} \\ 8x &= 11 && \color{red}{\text{Simplify both sides.}} \end{aligned}$$

Finally, to "undo" multiplying by 8, we divide both sides by 8.

$$8x \qquad 11$$

$$\begin{aligned} \frac{8x}{8} &= \frac{11}{8} && \text{Divide both sides by 8} \\ x &= \frac{11}{8} && \text{Simplify both sides.} \end{aligned}$$

Check: Let's use the TI-84 to check this solution.

1. First, store 11/8 in the variable X using the following keystrokes.

   $\boxed{1}\ \boxed{1}\ \boxed{\div}\ \boxed{8}\ \boxed{\text{STO}\triangleright}\ \boxed{\text{X,T},\theta,n}\ \boxed{\text{ENTER}}$

   The result is shown in the first image in Figure $2.2.5$.

2. Enter the left-hand side of the original equation: $2x-5(3-2x)$. Use the following keystrokes.

   $\boxed{2}\ \boxed{\times}\ \boxed{\text{X,T},\theta,n}\ \boxed{-}\ \boxed{5}\ \boxed{\times}\ \boxed{(}\ \boxed{3}\ \boxed{-}$

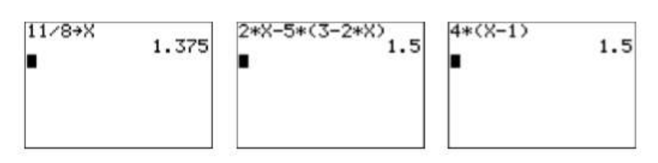
   $\boxed{2}\ \boxed{\times}\ \boxed{\text{X,T},\theta,n}\ \boxed{)}\ \boxed{\text{ENTER}}$

   The result is shown in the second image in Figure $2.2.5$.

3. Enter the right-hand side of the original equation: $4(x-1)$. Use the following keystrokes.

   $\boxed{4}\ \boxed{\times}\ \boxed{(}\ \boxed{\text{X,T},\theta,n}\ \boxed{-}\ \boxed{1}\ \boxed{)}\ \boxed{\text{ENTER}}$

   The result is shown in the third image in Figure \(\PageIndex{5}\).



```
11/8→X
             1.375
```

```
2*X-5*(3-2*X)
               1.5
```

```
4*(X-1)
               1.5
```

***Figure** $2.2.5$**:** Checking the solution to $2x-5(3-2x)=4(x-1)$.*

There is no need to use the 1:►Frac from the MATH menu this time. The fact that both sides of the equation evaluate to an identical 1.5 when $x=11/8$ guarantees that $11/8$ is a solution of $2x-5(3-2x)=4(x-1)$.

Exercise 2.2.6

Solve for $x: 5(1-x)=2(x+3)-(x-1)$.

Answer

$x=-1/3$

Contributors

- David Arnold (Retired Professor (Mathematics) at College of the Redwoods)

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2.3: Clearing Fractions and Decimals

In this section we introduce techniques that clear fractions and decimals from equations, making the resulting equation a lot easier to solve. When clearing fractions from an equation, you will need to simplify products like the ones posed in the following examples.

Example 2.3.1

Simplify: $12\left(\frac{2}{3}x\right)$.

Solution

When we multiply three numbers, such as 12, $2/3$, and x, the associative property of multiplication tells us that it does not matter which two numbers we multiply first. We use the associative property to regroup, then multiply numerators and denominators and simplify the result.

$$\begin{aligned} 12\left(\frac{2}{3}x\right) &= \left(12\cdot\frac{2}{3}\right)x \quad \text{Associative property of multiplication.} \\ &= \frac{24}{3}x \quad \text{Multiply: } 12\cdot 2 = 24 \\ &= 8x \quad \text{Divide: } 24/3 = 8 \end{aligned}$$

Exercise 2.3.1

Simplify: $15\left(\frac{3}{5}x\right)$.

Answer

$9x$

Example $2.3.1$ shows all of the steps involved in arriving at the answer. However, the goal in this section is to perform this calculation mentally. So we just “Multiply 12 and 2 to get 24, then divide 24 by 3 to get 8.” This approach allows us to write down the answer without doing any work.

$$12\left(\frac{2}{3}x\right) = 8x$$

You should practice this mental calculation until you can write down the answer without writing down any steps.

Example 2.3.2

Simplify: $18\left(\frac{2}{9}x\right)$.

Solution

This time we perform the calculations mentally. Multiply 18 and 2 to get 36, then divide 36 by 9 to get 4.

$$18\left(\frac{2}{9}x\right) = 4x$$

Exercise 2.3.2

Simplify: $14\left(\frac{3}{7}x\right)$.

Answer

$6x$

When the numbers get larger, the mental calculations become harder. For example, consider

$$72\left(\frac{8}{9}x\right)$$

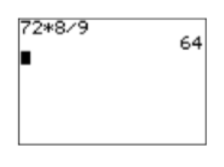
In this case, the work “multiply 72 and 8 to get 576, then divide 576 by 9 to get 64” is a bit difficult to carry in your head. However, this is when the calculator comes to the rescue.

Example $2.3.3$

Use your calculator to help simplify $72\left(\frac{8}{9}x\right)$.

Solution

Use your calculator to multiply 72 and 8, then divide by 9. Enter 72*8/9 and press the ENTER key.



```
72*8/9
                64
```

Thus, $72\left(\frac{8}{9}x\right) = 64x$.

Exercise $2.3.3$

Use your calculator to simplify: $81\left(\frac{5}{9}x\right)$.

Answer

$45x$

Canceling is more efficient

In Examples $2.3.1$, \(\PageIndex{2}\), and \(\PageIndex{3}\), we multiplied numerators, then divided by the sole denominator. We also saw that it is a bit difficult to carry the work in our head as the numbers grow larger. In Chapter 1, Section 3, we saw that canceling reduces the size of the numbers and simplifies the work.

Example $2.3.4$

Simplify: $72\left(\frac{8}{9}x\right)$.

Solution

In Example $2.3.3$, we used our calculator to multiply 72 and 8 to get 576, then divided 576 by 9 to get 64. In this solution, we divide 9 into 72 to get 8, then multiply 8 by 8 to get 64. We get the same answer, but because the intermediate numbers are much smaller, the calculations are much easier to do mentally.

$$\begin{aligned} 72\left(\frac{8}{9}x\right) &= \left(72\cdot\frac{8}{9}\right)x \quad \text{Associative property of multiplication} \\ &= (8\cdot 8)x \quad \text{Divide: } 72/9 = 8 \\ &= 64x \quad \text{Multiply: } 8\cdot 8 = 64 \end{aligned}$$

Exercise $2.3.4$

Simplify: $64\left(\frac{5}{8}x\right)$.

Answer

$40x$

Example $2.3.4$ shows all of the steps involved in arriving at the answer. Again, the goal in this section is to perform this calculation mentally, so we just “Divide 9 into 72 to get 8, then multiply 8 by 8 to get 644.”

$$72\left(\frac{8}{9}x\right) = 64x$$

Not only does this approach allow us to write down the answer without doing any work, the numerical calculations involve smaller numbers. You should practice this mental calculation until you can write down the answer without writing down any steps.

Example $2.3.5$

Simplify: $27\left(\frac{5}{9}x\right)$.

Solution

Divide 9 into 27 to get 3, then multiply 3 by 5 to get 15.

$$27\left(\frac{5}{9}x\right) = 15x$$

Exercise $2.3.5$

Simplify: $18\left(\frac{3}{2}x\right)$.

Answer

$27x$

Note

The technique shown in Examples \(\PageIndex{4}\) and \(\PageIndex{5}\) is the technique we'll use in the remainder of this section. Dividing (canceling) first is far more efficient, the smaller numbers allowing us to perform the calculation mentally.

Clearing Fractions from an Equation

Now that we've done the required fraction work, we can now concentrate on clearing fractions from an equation. Once the fractions are removed from the equation, the resulting equivalent equation is far easier to solve than the original.

Clearing fractions from an equation

To clear fractions from an equation, multiply both sides of the equation by the least common denominator.

Example $2.3.6$

Add text here.

Solution

Add text here.

Exercise $2.3.6$

Add exercises text here.

Answer

Add texts here. Do not delete this text first.

Contributors

- David Arnold (Retired Professor (Mathematics) at College of the Redwoods)

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2.4: Formulae

The formulae of science usually contain variable letters other than the variable "Formulae" is the plural for "formula."x . Indeed, formulae in science typically use several letters. For example, Isaac Newton's Universal Law of Gravitation says that the magnitude of the force of attraction between two celestial bodies is given by the formula:

$$F = \frac{GMm}{r^2}$$

where m usually denotes the mass of the smaller body, M the mass of the larger body, and r is the distance between the two bodies. The letter G represents the universal gravitational constant, having value $6.67428 \times 10^{-11}\ N(m/kg)^2$.

Variable case

Note the use of upper and lower case letter M's in Newton's Law of Gravitation. When working with scientific formulae, you must maintain the case of the given letters. You are not allowed to substitute lower for upper case, or upper for lower case in your work.

In Section 2.2, we described the goal that must be met when we are asked to "solve an equation for " x.

Solve for x

When asked to solve an equation for x, the goal is to manipulate the equation into the final form

$$x = \text{“Stuff”}$$

where "Stuff" is a valid mathematical expression that may contain other variables, mathematical symbols, etc., but it must not contain any occurrence of the variable x.

Thus, to solve an equation for x, we need to isolate the terms containing x on one side of the equation, and all remaining terms on the other side of the equation.

Example 2.4.1

Solve for $x : x + a = b$.

Solution

To undo the effects of adding a, subtract a from both sides of the equation.

$$\begin{aligned} x + a &= b && \text{Original equation.} \\ x + a - a &= b - a && \text{Subtract } a \text{ from both sides.} \\ x &= b - a && \text{Simplify.} \end{aligned}$$

Exercise 2.4.1

Solve for $x : x - c = d$.

Answer

$x = c + d$

In Example 2.4.1, note that the answer $x = b - a$ has the required form, $x = $ "Stuff", where "Stuff" is a valid mathematical expression that contains other variables, mathematical symbols, etc., but it does not contain any occurrence of the variable x. Now, what if we were asked to solve the same equation for a, instead of x?

Example 2.4.2

Solve for $a: x+a=b$.

Solution

We are instructed to solve the equation $x+a=b$ for a. This means that our final answer must have the form $a=\text{“Stuff”}$, where “Stuff” is a valid mathematical expression that contains other variables, mathematical symbols, etc., but it does not contain any occurrence of the variable a. This means that we must isolate all terms containing the variable a on one side of the equation, and all remaining terms on the other side of the equation. Now, to undo the effect of adding x, subtract x from both sides of the equation.

$$\begin{aligned} x+a&=b && \text{Original equation.} \\ x+a-x&=b-x && \text{Subtract } x \text{ from both sides.} \\ a&=b-x && \text{Simplify.} \end{aligned}$$

Note that we have $a=\text{“Stuff”}$, where “Stuff” contains no occurrence of a, the variable we are solving for.

Exercise 2.4.2

Solve for $x: x-c=d$.

Answer

$c=x-d$

Example 2.4.3

The formula $F=kx$, known as “Hooke’s Law”, predicts the force F required to stretch a spring x units. Solve the equation for k.

Solution

We are instructed to solve the equation $F=kx$ for k. This means that our final answer must have the form $k=\text{“Stuff”}$, where “Stuff” is a valid mathematical expression that may contain other variables, mathematical symbols, etc., but it may not contain any occurrence of the variable k. This means that we must isolate all terms containing the variable k on one side of the equation, and all remaining terms on the other side of the equation. However, note that all terms containing the variable k are already isolated on one side of the equation. Terms not containing the variable k are isolated on the other side of the equation. Now, to “undo” the effect of multiplying by x, divide both sides of the equation by x.

$$\begin{aligned} F&=kx && \text{Original equation.} \\ \frac{F}{x}&=\frac{kx}{x} && \text{Divide both sides by } x \\ \frac{F}{x}&=k && \text{Simplify.} \end{aligned}$$

Saying that $F/x=k$ is equivalent to saying that $k=F/x$. We can leave our answer in the form shown in the last step, but some instructors insist that we write the answer as follows:

$$k=\frac{F}{x} \quad \frac{F}{x}=k \text{ is equivalent to } k=\frac{F}{x}$$

Note that we have $k=\text{“Stuff”}$, where “Stuff” contains no occurrence of k, the variable we are solving for.

Exercise 2.4.3

Solve for $m: E=mc^2$.

Answer

$m=\frac{E}{c^2}$

Example 2.4.4

The formula $V = RI$ is called "Ohm's Law." It helps calculate the voltage drop V across a resistor R in an electric circuit with current I. Solve the equation for R.

Solution

We are instructed to solve the equation $V = RI$ for R. This means that our final answer must have the form $R = \text{“Stuff”}$, where "Stuff" is a valid mathematical expression that may contain other variables, mathematical symbols, etc., but it may not contain any occurrence of the variable R. This means that we must isolate all terms containing the variable R on one side of the equation, and all remaining terms on the other side of the equation. However, note that all terms containing the variable R are already isolated on one side of the equation. Terms not containing the variable R are isolated on the other side of the equation. Now, to "undo" the effect of multiplying by I, divide both sides of the equation by I.

$$\begin{aligned} V &= RI \quad \text{Original equation.} \\ \frac{V}{I} &= \frac{RI}{I} \quad \text{Divide both sides by } I \\ \frac{V}{I} &= R \quad \text{Simplify.} \end{aligned}$$

This can also be written in the following form:

$$R = \frac{V}{I} \quad V/I = R \text{ is equivalent to } R = V/I$$

Note that we have $R = \text{“Stuff”}$, where "Stuff" contains no occurrence of R, the variable we are solving for.

Exercise 2.4.4

Solve for $t : d = st$.

Answer

$$t = \frac{d}{s}$$

Clearing Fractions

If fractions occur in a formula, clear the fractions from the formula by multiplying both sides of the formula by the common denominator.

Example 2.4.5

The formula $K = \frac{1}{2}mv^2$ is used to calculate the kinetic energy K of a particle of mass m moving with velocity v. Solve the equation for m.

Solution

We're asked to solve the equation $K = (1/2)mv^2$ for m. First, clear the fractions by multiplying both sides by the common denominator.

$$\begin{aligned} K &= \frac{1}{2}mv^2 \quad \text{Original equation.} \\ 2(K) &= 2\left(\frac{1}{2}mv^2\right) \quad \text{Multiply both sides by } 2 \\ 2K &= mv^2 \quad \text{Simplify. Cancel } 2' \text{ s.} \end{aligned}$$

Note that all terms containing m, the variable we are solving for, are already isolated on one side of the equation. We need only divide both sides by v^2 to complete the solution.

$$\begin{aligned} \frac{2K}{v^2} &= \frac{mv^2}{v^2} \quad \text{Divide both sides by } v^2 \\ \frac{2K}{v^2} &= m \quad \text{Simplify. Cancel } v^2 \text{ for } v^2 \end{aligned}$$

Note that the final answer has the form $m =$ "Stuff", where "Stuff" contains no occurrence of the variable m.

Exercise 2.4.5

Solve for $g : s = \frac{1}{2}gt^2$.

Answer

$$g = \frac{2s}{t^2}$$

Example 2.4.6

As mentioned earlier, Newton's Universal Law of Gravitation is described by the formula

$$F = \frac{GMm}{r^2}$$

Solve this equation for m.

Solution

We're asked to solve the equation $F = GMm/r^2$ for m. First, clear the fractions by multiplying both sides by the common denominator.

$$\begin{aligned} F &= \frac{GMm}{r^2} && \text{Original equation.} \\ r^2(F) &= r^2\left(\frac{GMm}{r^2}\right) && \text{Multiply both sides by } r^2 \\ r^2F &= GMm && \text{Simplify. Cancel } r^2 \text{ for } r^2 \end{aligned}$$

Note that all terms containing m, the variable we are solving for, are already isolated on one side of the equation. We need only divide both sides by GM to complete the solution.

$$\begin{aligned} \frac{r^2F}{GM} &= \frac{GMm}{GM} && \text{Divide both sides by } GM \\ \frac{r^2F}{GM} &= m && \text{Simplify. Cancel } GM \text{ for } GM \end{aligned}$$

Note that the final answer has the form $m =$ "Stuff", where "Stuff" contains no occurrence of the variable m.

Exercise 2.4.6

Solve for $q_2 : F = \frac{kq_1q_2}{r^2}$.

Answer

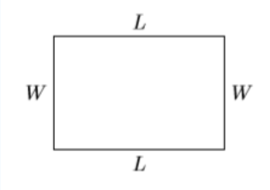
$$q_2 = \frac{Fr^2}{kq_1}$$

Geometric Formulae

Let's look at a few commonly used formulae from geometry.

Example 2.4.7

Let W and L represent the width and length of a rectangle, respectively, and let P represent its perimeter.



The perimeter (distance around) of the rectangle is found by summing its four sides, then combining like terms.

$$\begin{aligned} P &= L+W+L+W \quad \text{Summing the four sides.} \\ P &= 2W+2L \quad \text{Combine like terms.} \end{aligned}$$

Solve $P = 2W + 2L$ for L. Then, given that the perimeter is 300 feet and the width is 50 feet, use your result to calculate the length.

Solution

We're first asked to solve $P = 2W + 2L$ for L. First, isolate all terms containing the variable L on one side of the equation.

$$\begin{aligned} P &= 2W+2L \quad \text{Original equation.} \\ P-2W &= 2W+2L-2W \quad \text{Subtract } 2W \text{ from both sides.} \\ P-2W &= 2L \quad \text{Simplify.} \\ \frac{P-2W}{2} &= \frac{2L}{2} \quad \text{Divide both sides by 2} \\ \frac{P-2W}{2} &= L \quad \text{Simplify.} \end{aligned}$$

Note that the final result has $L =$ "Stuff", where "Stuff" contains no occurrence of the variable L.

The second part of this example requests that we find the length of the rectangle, given that the perimeter is $P = 300$ feet and the width is $W = 50$ feet. To calculate the length, substitute $P = 300$ and $W = 50$ in $L = (P - 2W)/2$.

$$\begin{aligned} L &= \frac{P-2W}{2} \quad \text{Perimeter formula solved for } L \\ L &= \frac{300-2(50)}{2} \quad 300 \text{ for } P, 50 \text{ for } W \\ L &= \frac{300-100}{2} \quad \text{Multiply: } 2(50)=100 \\ L &= \frac{200}{2} \quad \text{Subtract: } 300-100=200 \\ L &= 100 \quad \text{Divide: } 200/2=100 \end{aligned}$$

Hence, the length of the rectangle is 100 feet.

Exercise 2.4.7

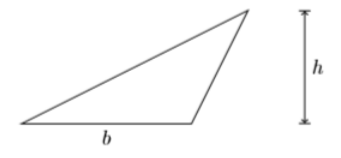
The perimeter of a rectangle is 160 meters and its width is 30 meters. Finds its length.

Answer

$L = 50$ meters

Example 2.4.8

Let b and h represent the length of the base and the height of a triangle, respectively, and let A represent the area of the triangle.



The area of the triangle is computed using the formula: $A = \frac{1}{2}bh$That is, the area of a triangle is "one-half the base times the height."

Solve the formula $A=\frac{1}{2}bh$ for h. Secondly, given that the area is $A=90$ in^2 (90 square inches) and the length of the base is 15 in (15 inches), find the height of the triangle.

Solution

We're first asked to solve $A=(1/2)bh$ for h. Because the equation has fractions, the first step is to clear the fractions by multiplying both sides by the least common denominator.

$$\begin{aligned} A &= \frac{1}{2}bh & \text{Area of a triangle formula.} \\ 2(A) &= 2\left(\frac{1}{2}bh\right) & \text{Multiply both sides by 2} \\ 2A &= bh & \text{Simplify.} \end{aligned}$$

Now, we already have all terms containing the variable h on one side of the equation, so we can solve for h by dividing both sides of the equation by b.

$$\begin{aligned} \frac{2A}{b} &= \frac{bh}{b} & \text{Divide both sides by } b \\ \frac{2A}{b} &= h & \text{Simplify.} \end{aligned}$$

Note that the final result has $h=$ "Stuff", where "Stuff" contains no occurrence of the variable h.

The second part of this example requests that we find the height of the triangle, given that the area is $A=90$ in 2 and the length of the base is $b=15$ in. To calculate the height of the triangle, substitute $A=90$ and $b=15$ in $h=2A/b$.

$$\begin{aligned} h &= \frac{2A}{b} & \text{Area formula solved for } h \\ h &= \frac{2(90)}{15} & \text{Substitue 90 for } A, 15 \text{ for } b \\ h &= \frac{180}{15} & \text{Multiply: } 2(90)=180 \\ h &= 12 & \text{Divide: } 180/15=12 \end{aligned}$$

Hence, the height of the triangle is 12 inches.

Exercise $2.4.8$

The area of a triangle is 140 cm^2 and the length of its base is 70 cm. Find the height of the triangle. .

Answer

4 cm

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2.5: Applications

The solution of a word problem must incorporate each of the following steps.

Requirements for Word Problem Solutions

1. Set up a Variable Dictionary. You must let your readers know what each variable in your problem represents. This can be accomplished in a number of ways:
    - Statements such as “Let P represent the perimeter of the rectangle.”
    - Labeling unknown values with variables in a table.
    - Labeling unknown quantities in a sketch or diagram.
2. Set up an Equation.Every solution to a word problem must include a carefully crafted equation that accurately describes the constraints in the problem statement.
3. Solve the Equation. You must always solve the equation set up in the previous step.
4. Answer the Question. This step is easily overlooked. For example, the problem might ask for Jane’s age, but your equation’s solution gives the age of Jane’s sister Liz. Make sure you answer the original question asked in the problem. Your solution should be written in a sentence with appropriate units.
5. Look Back. It is important to note that this step does not imply that you should simply check your solution in your equation. After all, it’s possible that your equation incorrectly models the problem’s situation, so you could have a valid solution to an incorrect equation. The important question is: “Does your answer make sense based on the words in the original problem statement.”

Let’s give these requirements a test drive.

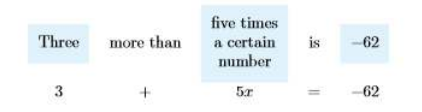
Example 2.5.1

Three more than five times a certain number is -62. Find the number.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let x represent the unknown number.
2. *Set up an Equation*.“Three more than five times a certain number is -62” becomes:



Three	more than	five times a certain number	is	-62
3	$+$	$5x$	$=$	-62

3. *Solve the Equation*. To solve for x, first subtract 3 from both sides of the equation.

$$\begin{aligned} 3+5x &= -62 && \text{Original equation.} \\ 3+5x-3 &= -62-3 && \text{Subtract 3 from both sides.} \\ 5x &= -65 && \text{Simplify.} \\ \frac{5x}{5} &= \frac{-65}{5} && \text{Divide both sides by 5} \\ x &= -13 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question*. The unknown number is -13.
5. *Look Back*. Compute “three more than five times -13.”

$$\begin{aligned} 3+5(-13) &= 3+(-65) \\ &= -62 \end{aligned}$$

Hence, three more than five times -13 is -62, as required. Our solution is correct.

Exercise 2.5.1

27 more than 5 times a certain number is -148. What is the number?

Answer

-35

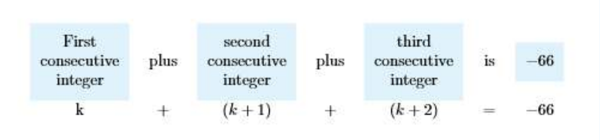
Example 2.5.2

The sum of three consecutive integers is -66. Find the smallest of these three integers.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let k represent the smallest of three consecutive integers.
2. *Set up an Equation*. An example of three consecutive integers is 34, 35, and 36. These are not the integers we seek, but they serve to help in the understanding of the problem. Note how each consecutive integer is one larger than the preceding integer. If k is the smallest of three consecutive integers, then the next two consecutive integers are $k+1$ and $k+2$. The "sum of three consecutive integers is -66" becomes:



First consecutive integer	plus	second consecutive integer	plus	third consecutive integer	is	-66
k	$+$	$(k+1)$	$+$	$(k+2)$	$=$	-66

3. *Solve the Equation*. To solve for k, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned} k+(k+1)+(k+2) &= -66 && \text{Original equation.} \\ 3k+3 &= -66 && \text{Combine like terms.} \\ 3k+3-3 &= -66-3 && \text{Subtract 3 from both sides.} \\ 3k &= -69 && \text{Simplify.} \\ \frac{3k}{3} &= \frac{-69}{3} && \text{Divide both sides by 3.} \\ k &= -23 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question*. The smallest of three consecutive integers is -23.
5. Look Back. If $k=-23$ is the smallest of three consecutive integers, then the next two consecutive integers are -22 and -21. Let's check the sum of these three consecutive integers.

$$-23+(-22)+(-21)=-66$$

Hence, the sum of the three consecutive integers is -66, as required. Our solution is correct

Exercise 2.5.2

The sum of three consecutive odd integers is -225. What are the integers?

Answer

$-77, -75, -73$

Example 2.5.3

A carpenter cuts a board measuring 60 inches in three pieces. The second piece is twice as long as the first piece, and the third piece is three times as long as the first piece. Find the length of each piece cut by the carpenter.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let L represent the length of the first piece. Then the second piece, which is twice as long as the first piece, has length $2L$. The third piece, which is three times as long as the first piece, has length $3L$. Let's construct a little table to help summarize the information provided in this problem.

Piece	Length (in)
First piece	L
Second piece	2L
Third piece	3L
Total length	60

2. *Set up an Equation*. As you can see in the table above, the second column shows that the sum of the three pieces is 60 inches. In symbols:

$$L + 2L + 3L = 60$$

3. *Solve the Equation*. To solve for L, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned} L + 2L + 3L &= 60 && \text{Original equation.} \\ 6L &= 60 && \text{Combine like terms.} \\ \frac{6L}{6} &= \frac{60}{6} && \text{Divide both sides by 6} \\ L &= 10 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question*. The first piece has length L = 10 inches. The second piece has length 2L = 20 inches. The third piece has length 3L = 30 inches. In tabular form, this is even more apparent.

Piece	Length (in)	Length(in)
First piece	L	10
Second piece	2L	20
Third piece	3L	30
Total length	60	60

5. *Look Back*. Not only is the second length twice the first and the third length three times the first, check the sum of their lengths:

$$10 + 20 + 30 = 60$$

That's a total of 60 inches. We have the correct solution.

Exercise 2.5.3

Han cuts a board measuring 230 inches in three pieces. The second piece is twice as long as the first piece, and the third piece is 30 inches longer than the second piece. Find the length of each piece cut by Han.

Answer

$40, 80, 110$ in

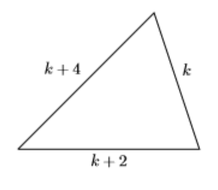
Example 2.5.4

The three sides of a triangle are consecutive even integers. If the perimeter (sum of the three sides) of the triangle is 156 centimeters, find the length of each side of the triangle.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. An example of three consecutive even integers is 18, 20, and 22. These are not the integers we seek, but they do give us some sense of the meaning of three consecutive even integers. Note that each consecutive even integer is two larger than the preceding integer. Thus, if k is the length of the first side of the triangle, then the next two sides are $k+2$ and $k+4$. In this example, our variable dictionary will take the form of a well-labeled figure.



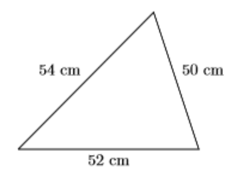
2. *Set up an Equation*. The perimeter of the triangle is the sum of the three sides. If the perimeter is 156 centimeters, then:

$$k+(k+2)+(k+4)=156$$

3. *Solve the Equation*. To solve for k, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned} k+(k+2)+(k+4)&=156 && \text{Original equation.}\\ 3k+6&=156 && \text{Combine like terms.}\\ 3k+6-6&=156-6 && \text{Subtract 6 from both sides.}\\ 3k&=150 && \text{Simplify.}\\ \frac{3k}{3}&=\frac{150}{3} && \text{Divide both sides by 3.}\\ k&=50 && \text{Simplify.} \end{aligned}$$

4. *Answer the Question*. Thus, the first side has length 50 centimeters. Because the next two consecutive even integers are $k+2=52$ and $k+4=54$, the three sides of the triangle measure 50, 52, and 54 centimeters, respectively.
5. *Look Back*. An image helps our understanding. The three sides are consecutive even integers.



Note that the perimeter (sum of the three sides) is:

$$50\text{cm}+52\text{cm}+54\text{cm}=156\text{cm}$$

Thus, the perimeter is 156 centimeters, as it should be. Our solution is correct.

Exercise 2.5.4

The three sides of a triangle are consecutive integers. If the perimeter (sum of the three sides) of the triangle is 453 centimeters, find the length of each side of the triangle.

Answer

$150, 151, 152$ cm

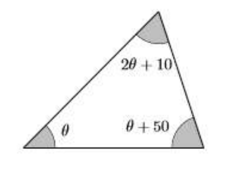
Example 2.5.5

A well-known fact from geometry is the fact that the sum of the angles of a triangle is 180°. Suppose we have a triangle whose second angle is 10 degrees larger than twice its first angle and whose third angle is 50 degrees larger than its first angle. Find the measure of each angle of the triangle.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. The Greek alphabet starts out with the letters $\alpha, \beta, \gamma, \delta, \epsilon, \ldots$, in much the same way that the English alphabet start out with the letters $a, b, c, d, e, \ldots$ Mathematicians love to use Greek letters, especially in the study of trigonometry. The greek letter θ (pronounced "theta") is particularly favored in representing an angle of a triangle. So, we'll let θ represent the degree measure of the first angle of the triangle. The second angle is 10 degrees larger than twice the first angle, so the second angle is $2\theta + 10$. The third angle is 50 degrees larger than the first angle, so the third angle is $\theta + 50$. Again, we'll set up a well-labeled figure for our variable dictionary.



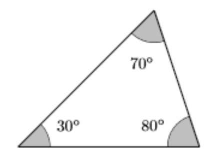
2. *Set up an Equation*. The sum of the angles is 180°, so:

$$\theta + (2\theta + 10) + (\theta + 50) = 180$$

3. *Solve the Equation*. To solve for θ, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned}
\theta + (2\theta + 10) + (\theta + 50) &= 180 \quad \color{red}{\text{Original equation.}} \\
4\theta + 60 &= 180 \quad \color{red}{\text{Combine like terms.}} \\
4\theta + 60 - 60 &= 180 - 60 \quad \color{red}{\text{Subtract 60 from both sides.}} \\
4\theta &= 120 \quad \color{red}{\text{Simplify.}} \\
\frac{4\theta}{4} &= \frac{120}{4} \quad \color{red}{\text{Divide both sides by 4.}} \\
\theta &= 30 \quad \color{red}{\text{Simplify.}}
\end{aligned}$$

4. *Answer the Question*. Thus, the first angle is $\theta = 30$ degrees, the second angle is $2\theta + 10 = 70$ degrees, and the third angle is $\theta + 50 = 80$ degrees.
5. *Look Back*. An image helps our understanding. Note that the second angle is 10 degrees larger than twice the first angle. Note that the third angle is 50 degrees larger than the first angle.



Note that the sum of the angles is:

$$30^\circ + 70^\circ + 80^\circ = 180^\circ$$

Thus, the sum of the three angles is 180 degrees, as it should be. We have the correct solution.

Exercise 2.5.5

The second angle of a triangle is three times bigger than the first angle. The third angle of the triangle is a 40 degrees larger than the second angle. How many degrees are in each angle?

Answer

$20^\circ, 60^\circ, 100^\circ$

Example 2.5.6

Martha inherits $\$21,000$ and decides to invest the money in three separate accounts. The amount she invests in the second account is twice the amount she invests in the first account. The amount she invests in the third account is $\$1,000$ more than the amount she invests in the second account. How much did she invest in each account?

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. We'll use a table in this example to help set up our variable dictionary. Let x be the amount invested in the first account. The amount invested in the second account is twice that invested in the first account, so $2x$ is the amount invested in the second account. The third account investment is $\$1,000$ more than the amount invested in the second account, so that is $2x+1000$.

Account #	Amount Invested
Account#1	x
Account#2	2x
Account#3	2x+1000
Total Invested	21000

2. *Set up an Equation*. The second column of the table reveals the required equation. The three investments must sum to $\$21,000$.

$$x+2x+(2x+1000)=21000$$

3. *Solve the Equation*. To solve for x, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned}
x+2x+(2x+1000) &= 21000 && \text{Original equation.}\\
5x+1000 &= 21000 && \text{Combine like terms.}\\
5x+1000-1000 &= 21000-1000 && \text{Subtract 1000 from both sides.}\\
5x &= 20000 && \text{Simplify.}\\
\frac{5x}{5} &= \frac{20000}{5} && \text{Divide both sides by 5.}\\
x &= 4000 && \text{Simplify.}
\end{aligned}$$

4. *Answer the Question*. Substitute $x=4000$ in each entry of the second column of the table above to produce the results in the table below.

Account #	Amount Invested	Amount Invested
Account#1	x	$4,000
Account#2	2x	$8,000
Account#3	2x+1000	$9,000
Total Invested	21000	$21,000

5. Look Back. As we can see in our answer table, the amount $\$8,000$ invested in the second account is twice the amount invested in the first account. The amount $\$9,000$ invested in the third account is $\$1,000$ more than the amount invested in the second account. Moreover, the total investment is:

$$\$4,000+\$8,000+\$9,000=\$21,000$$

Thus, the total investment is $\$21,000$, as it should be. We have the correct solution.

Exercise $2.5.6$

Jim inherits $\$15,000$. He invests part in a fund that pays 5% per year and the rest in a fund that pays 4% per year. At the end of one year, the combined interest from both investments was $\$4,250$. How much did he invest in each fund?

Answer

$\$5,000$ at 5% and $\$10,000$ at 4%.

Example $2.5.7$

Jeff is hiking the $2,650$-mile Pacific Crest Trail from Mexico to Canada. Shortly before he crosses over from Oregon into Washington he is four times as far from the beginning of the trail as he is from the end. How much further does he have to hike?

Solution

In the solution, we address each step of the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary*. Let d represent the distance left for Jeff to hike. Because Jeff is four times further from the beginning of the trail than the end, the distance Jeff already completed is $4d$. Let's construct a little table to help summarize the information provided in this problem.

Section of Trail	Distance (mi)
Distance to finish	d
Distance from start	4d
Total distance	2650

2. *Set up an Equation*. As you can see in the table above, the second column shows that the sum of the two distances is 2650 miles. In symbols: \[d+4 d=2650 \nonumber \
3. *Solve the Equation*. To solve for d, first simplify the left-hand side of the equation by combining like terms.

$$\begin{aligned} d+4d &= 2650 && \color{red}{\text{Original equation.}} \\ 5d &= 2650 && \color{red}{\text{Combine like terms.}} \\ \frac{5d}{5} &= \frac{2650}{5} && \color{red}{\text{Divide both sides by 5}} \\ d &= 530 && \color{red}{\text{Simplify.}} \end{aligned}$$

4. *Answer the Question*. Jeff still has 530 miles to hike.
5. *Look Back*. Because the amount left to hike is $d=530$ miles, Jeff's distance from the start of the trail is $4d=4(530)$, or $2,120$ miles. If we arrange these results in tabular form, it is evident that not only is the distance from the start of the trail four times that of the distance left to the finish, but also the sum of their lengths is equal to the total length of the trail.

Section of Trail	Distance (mi)	Distance (mi)
Distance to finish	d	530

Section of Trail	Distance (mi)	Distance (mi)
Distance from start	4d	2120
Total distance	2650	2650

Thus, we have the correct solution.

Exercise $2.5.7$

Margaret is cycling along a lane that measures 100 miles. If Magaret is four times as far from the start of the ride as she is from the finish, how many more miles does she have to go before she finishes her ride?

Answer

20 miles

Example $2.5.8$

Today 15% of Sister Damaris' seventh grade class were ill and stayed home from school. If only 34 students are present, what is the actual size of Sister Damaris' class?

Solution

In the solution, we address each step of the Requirements for Word Problem Solutions.

1. *Set up a Variable Dictionary*. Let S represent the actual size of Sister Damaris' class.
2. Set up an Equation. If 15% of Sister Damaris' class was absent, then 85% of her class was present. There are 34 student present, so the phrase "85% of Sister Damaris' class is 34" translates into the equation,

$$0.85S = 34$$

   where we've changed 85% to a decimal by moving the decimal point two places to the left.
3. Solve the Equation. To solve for S, first clear the decimals by multiplying both sides of the equation by 100.

$$\begin{aligned} 0.85S &= 34 & \color{red}{\text{Original equation.}} \\ 85S &= 3400 & \color{red}{\text{Multiply both sides by 100.}} \\ \frac{85S}{85} &= \frac{3400}{85} & \color{red}{\text{Divide both sides by 85}} \\ S &= 40 & \color{red}{\text{Simplify.}} \end{aligned}$$

4. *Answer the Question*. Sister Damaris' class size is 40.
5. *Look Back*. We're told that 15% of Sister Damaris' class is absent. If we calculate 15% of 40, we get:

$$0.15(40) = 6$$

   Thus, there were 6 students absent, so $40 - 6$, or 34 students were present. Thus, we have the correct solution.

Exercise $2.5.8$

20% of Mary's class were ill and stayed home from school. If only 36 students are present, what is the actual size of Mary's class?

Answer

45

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2.6: Inequalities

In Chapter 1, we introduced the natural numbers $\mathbb{N}=\{1,2,3,\ldots\}$, the whole numbers $\mathbb{W}=\{0,1,2,3,\ldots\}$, and the integers $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. Later in the chapter, we introduced the rational numbers, numbers of the form p/q, where p and q are integers. We noted that both terminating and repeating decimals are rational numbers. Each of these numbers has a unique position on the number line (see Figure $2.6.1$).

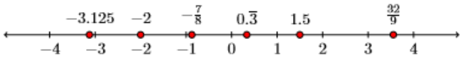


Figure $2.6.1$: Positioning numbers on the number line.

The natural numbers, whole numbers, and integers are also rational numbers, because each can be expressed in the form $\frac{p}{q}$, where p and q are integers. For example, $0=\frac{0}{12}$, $4=\frac{4}{1}$, and $-3=\frac{-12}{4}$. Indeed, the rational numbers contain all of the numbers we've studied up to this point in the course. However, not all numbers are rational numbers. For example, consider the decimal number $-3.10110111011110\ldots$, which neither terminates nor repeats. The number $\sqrt{2}=1.414213562373095\ldots$ also equals a decimal number that never terminates and never repeats. A similar statement can be made about the number $\pi=3.141592653589793\ldots$ Each of these irrational (not rational) numbers also has a unique position on the number line (see Figure $2.6.2$).

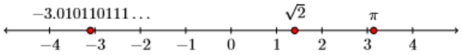


Figure $2.6.2$: Positioning numbers on the number line.

> Two other irrational numbers you may encounter in your mathematical studies are e (Euler's constant), which is approximately equal to $e\approx 2.71828182845904\ldots$, and ϕ (pronounced "phi"), called the golden ratio, which equals $\phi=\frac{1+\sqrt{5}}{2}$. The number e arises in applications involving compound interest, probability, and other areas of mathematics. The number ϕ is used in financial markets and is also arguably the ratio of beauty in art and architecture.

The Real Numbers

If we combine all of the rational and irrational numbers into one collection, then we have a set of numbers that is called the set of real numbers. The set of real numbers is denoted by the symbol $\mathbb{R}$.

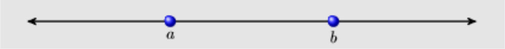
Every point on the number line is associated with a unique real number. Conversely, every real number is associated with a unique position on the number line. In lieu of this correspondence, the number line is usually called the real line.

Ordering the Real Numbers

The real numbers are ordered on the real line in a manner identical to how we ordered the integers on the number line in Section 1 of Chapter 1.

Order on the Real Line

Suppose that a and b are real numbers positioned on the real line as shown below.



- Because a lies to the "left of" b, we say that a is "less than" b, or in mathematical symbols, $a<b$. The inequality symbol $<$ is read "less than."
- Alternately, b lies to the "right of" a, so we can also say that b is "greater than" a, or in mathematical symbols, $b>a$. The inequality symbol $>$ is read "greater than."

Here are two more inequality symbols that we will use in this section.

Less than or equal to

If we want to say that a lies to the “left of” b, or shares the same position as b, then we say that a is “less than or equal to” b and write $a \leq b$. The inequality symbol $\leq$ is pronounced “less than or equal to.”

Greater than or equal to

If we want to say that b lies to the “right of” a, or shares the same position as a, then we say that b is “greater than or equal to" a and write $b \geq a$. The inequality symbol $\geq$ is pronounced “greater than or equal to.”

Set-Builder Notation

Mathematicians use a construct called set-builder notation to describe sets or collections of numbers. The general form of set-builder notation looks as follows:

$$\{x : \text{ some statement about } x\}$$

For example, suppose that we want to describe the set of “all real numbers that are less than 2.” We could use the following notation:

$$A=\{x : x<2\}$$

This is read aloud as follows: “A equals the set of all x such that x is less than 2.” Some prefer to use a vertical bar instead of a colon.

$$A=\{x \mid x<2\}$$

In this text we use the colon in set-builder notation, but feel free to use the vertical bar instead. They mean the same thing. One might still object that the notation

$$\{x : x<2\}$$

is a bit vague. One objection could be “What type of numbers x are you referring to? Do you want the integers that are less than two or do you want the real numbers that are less than two?” As you can see, this is a valid objection. One way of addressing this objection is to write:

$$A=\{x \in \mathbb{R} : x<2\} \quad \text{ or } \quad A=\{x \in \mathbb{N} : x<2\}$$

The first is read “A is the set of all x in R that are less than two,” while the second is read “A is the set of all x in N that are less than two.”

Set-builder Assumption

In this text, unless there is a specific reference to the set of numbers desired, we will assume that the notation $\{x : x<2\}$ is asking for the set of all real numbers less than 2.

In Figure $2.6.3$, we’ve shaded the set of real numbers $\{x : x<2\}$. Because

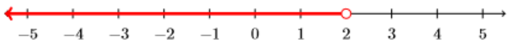


Figure $2.6.3$: Shading the numbers less than 2.

“less than” is the same as saying “left of,” we’ve shaded (in red) all points on the real line that lie to the left of the number two. Note that there is an “empty circle” at the number two. The point representing the number two is not shaded because we were only asked to shade the numbers that are strictly less than two.

While the shading in Figure $2.6.3$ is perfectly valid, much of the information provided in Figure $2.6.3$ is unnecessary (and perhaps distracting). We only need to label the endpoint and shade the real numbers to the left of two, as we’ve done in constructing Figure $2.6.4$.

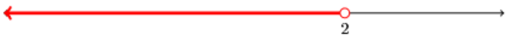


Figure $2.6.4$: You only need to label the endpoint.

For contrast, suppose instead that we're asked to shade the set of real numbers $\{x : x \leq 2\}$. This means that we must shade all the real numbers

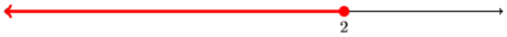


Figure $2.6.5$: You only need to label the endpoint.

that are "less than or equal to 2" or "left of and including 2." The resulting set is shaded in Figure $2.6.5$.

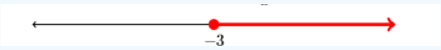
Note the difference between Figures $2.6.4$ and $2.6.45$. In Figures $2.6.4$ we're shading the set $\{x : x < 2\}$, so the number 2 is left unshaded (an empty dot). In Figures $2.6.5$, we're shading the set $\{x : x \leq 2\}$, so the number 2 is shaded (a filled-in dot).

Example $2.6.1$

Shade the set $\{x : x \geq -3\}$ on the real line.

Solution

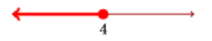
The notation $\{x : x \geq -3\}$ is pronounced "the set of all real numbers x such that x is greater than or equal to -3." Thus, we need to shade the number -3 and all real numbers to the right of -3.



Exercise $2.6.1$

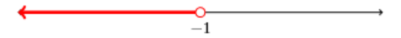
Shade $\{x : x \leq 4\}$ on the real line.

Answer



Example $2.6.2$

Use set-builder notation to describe the set of real numbers that are shaded on the number line below.



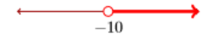
Solution

The number -1 is not shaded. Only the numbers to the left of -1 are shaded. This is the set of all real numbers x such that x is "less than" -1. Thus, we describe this set with the following set-builder notation:

$$\{x : x < -1\}$$

Exercise $2.6.2$

Use set-builder notation to describe the following set of real numbers:

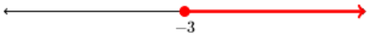


Answer

$\{x : x > -10\}$

Interval Notation

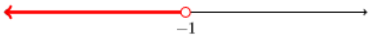
In Examples $2.6.1$ and $2.6.2$, we used set-builder notation to describe the set of real numbers greater than or equal to -3 and a second set of real numbers less than -1. There is another mathematical symbolism, called interval notation, that can be used to describe these sets of real numbers. Consider the first set of numbers from Example $2.6.1$, $\{x : x \geq -3\}$.



Sweeping our eyes "from left to right", we use $[-3, \infty)$ to describe this set of real numbers. Some comments are in order:

1. The bracket at the left end means that -3 is included in the set.
2. As you move toward the right end of the real line, the numbers grow without bound. Hence, the ∞ symbol (positive infinity) is used to indicate that we are including all real numbers to the right of -3. However, ∞ is not really a number, so we use a parentheses to indicate we are "not including" this fictional point.

The set of numbers from Example $2.6.1$ is $\{x : x < -1\}$.



Sweeping our eyes "from left to right", this set of real numbers is described with $(-\infty, -1)$. Again, comments are in order:

1. The number -1 is not included in this set. To indicate that it is not included, we use a parenthesis.
2. As you move toward the left end of the real line, the numbers decrease without bound. Hence, the $-\infty$ symbol (negative infinity) is used to indicate that we are including all real numbers to the left of -1. Again, $-\infty$ is not an actual number, so we use a parenthesis to indicate that we are not including this "fictional" point.

Sweep your eyes from "left to right"

If you would like to insure that you correctly use interval notation, place the numbers in your interval notation in the same order as they are encountered as you sweep your eyes from "left to right" on the real line.

A nice summary of set-builder and interval notation is presented in Table $2.6.1$ at the end of the section.

Equivalent Inequalities

Like equations, two inequalities are equivalent if they have the same solution sets.

Adding or Subtracting the Same Quantity from Both Sides of an Inequality

Let a and b be real numbers with

$$a < b$$

If c is any real number, then

$$a + c < b + c$$

and

$$a - c < b - c$$

That is, adding or subtracting the same amount from both sides of an inequality produces an equivalent inequality (does not change the solution).

Example $2.6.3$

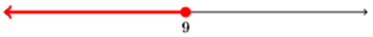
Solve for x : $x - 2 \leq 7$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To "undo" subtracting 2, we add 2 to both sides of the inequality.

$$\begin{aligned} x - 2 &\leq 7 && \text{Original inequality.} \\ x - 2 + 2 &\leq 7 + 2 && \text{Add 2 to both sides.} \\ x &\leq 9 && \text{Simplify both sides.} \end{aligned}$$

To shade the real numbers less than or equal to 9, we shade the number 9 and all real numbers to the left of 9.



Using set-builder notation, the solution is $\{x : x \leq 9\}$. Using interval notation, the solution is $(-\infty, 9]$.

Exercise 2.6.3

Use interval notation to describe the solution of: $x - 7 < -8$.

Answer

$(-\infty, -1)$

If we multiply or divide both sides of an inequality by a positive number, we have an equivalent inequality.

Multiplying or Dividing by a Positive Number

Let a and b be real numbers with $a < b$. If c is a real positive number, then

$$ac < bc$$

and

$$\frac{a}{c} < \frac{b}{c}$$

Example 2.6.4

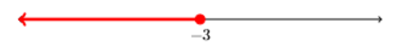
Solve for x : $\quad 3x \leq -9$ Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To “undo” multiplying by 3, divide both sides of the inequality by 3. Because we are dividing both sides by a positive number, we do not reverse the inequality sign.

$$\begin{aligned} 3x &\leq -9 && \text{Original inequality.} \\ \frac{3x}{3} &\leq \frac{-9}{3} && \text{Divide both sides by 3.} \\ x &\leq -3 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers less than or equal to -3.



Using set-builder notation, the solution is $\{x : x \leq -3\}$. Using interval notation, the solution is $(-\infty, -3]$.

Exercise 2.6.4

Use interval notation to describe the solution of:

$$2x > -8$$

Answer

$(-4, \infty)$

Reversing the Inequality Sign

Up to this point, it seems that the technique for solving inequalities is pretty much identical to the technique used to solve equations. However, in this section we are going to encounter one exception.

Suppose we start with the valid inequality $-2 < 5$, then we multiply both sides by 2, 3, and 4.

$$\begin{array}{rrr} -2 < 5 & -2 < 5 & -2 < 5 \\ 2(-2) < 2(5) & 3(-2) < 3(5) & 4(-2) < 4(5) \\ -4 < 10 & -6 < 15 & -8 < 20 \end{array}$$

Note that in each case, the resulting inequality is still valid.

Caution! We're about to make an error!

Start again with $-2 < 5$, but this time multiply both sides by -2, -3, and -4.

$$\begin{array}{rrr} -2 < 5 & -2 < 5 & -2 < 5 \\ -2(-2) < -2(5) & -3(-2) < -3(5) & -4(-2) < -4(5) \\ 4 < -10 & 6 < -15 & 8 < -20 \end{array}$$

> In each of the resulting inequalities, the inequality symbol is pointing the wrong way!

When you multiply both sides of an inequality by a negative number, you must reverse the inequality sign. Starting with $-2 < 5$, multiply both sides by -2, -3, and -4, but reverse the inequality symbol.

> Some readers might prefer a more formal reason as to why we reverse the inequality when we multiply both sides by a negative number. Suppose that $a < b$. Then, subtracting b from both sides gives the result $a - b < 0$. This means that $a - b$ is a negative number. Now, if c is a negative number, then the product $(a-b)c$ is positive. Then:
>
> $$\begin{aligned} (a-b)c &> 0 \\ ac - bc &> 0 \\ ac - bc + bc &> 0 + bc \\ ac &> bc \end{aligned}$$
>
> Thus, if you start with $a < b$ and $c < 0$, then $ac > bc$.

Multiplying or Dividing by a Negative Number

Let a and b be real numbers with $a < b$. If c is a real negative number, then

$$ac > bc$$

and

$$\frac{a}{c} > \frac{b}{c}$$

That is, when multiplying or dividing both sides of an inequality by a negative number, you must **reverse** the inequality sign.

Example 2.6.5

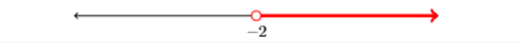
Solve for $x: \quad -2x < 4$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

To "undo" multiplying by -2, divide both sides by -2. Because we are dividing both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned} -2x &< 4 && \text{Original inequality.} \\ \frac{-2x}{-2} &> \frac{4}{-2} && \text{Divide both sides by } -2 \\ x &> -2 && \text{Reverse the inequality sign.} \\ x &> -2 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers greater than -2.



Using set-builder notation, the solution is $\{x : x > -2\}$. Using interval notation, the solution is $(-2, \infty)$.

Exercise 2.6.5

Use interval notation to describe the solution of:

$$-3x \geq -6$$

Answer

$(-\infty, 2]$

Multiple Steps

Sometimes you need to perform a sequence of steps to arrive at the solution.

Example 2.6.6

Solve for x : $2x + 5 > -7$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

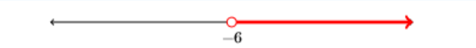
To “undo” adding 5, subtract 5 from both sides of the inequality.

$$\begin{aligned} 2x+5 &> -7 && \text{Original inequality.} \\ 2x+5-5 &> -7-5 && \text{Subtract 5 from both sides.} \\ 2x &> -12 && \text{Simplify both sides.} \end{aligned}$$

To “undo” multiplying by 2, divide both sides by 2. Because we are dividing both sides by a positive number, we do not reverse the inequality sign.

$$\begin{aligned} \frac{2x}{2} &> \frac{-12}{2} && \text{Divide both sides by 2} \\ x &> -6 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers greater than -6.



Using set-builder notation, the solution is $\{x : x > -6\}$. Using interval notation, the solution is $(-6, \infty)$.

Exercise 2.6.6

Use interval notation to describe the solution of:

$$3x - 2 \leq 4$$

Answer

$(-\infty, 2]$

Example 2.6.7

Solve for x : $3 - 5x \leq 2x + 17$. Sketch the solution on the real line, then use set-builder and interval notation to describe your solution.

Solution

We need to isolate terms containing x on one side of the inequality. Start by subtracting $2x$ from both sides of the inequality.

$$\begin{aligned} 3-5x &\leq 2x+17 && \text{Original inequality.} \\ 3-5x-2x &\leq 2x+17-2x && \text{Subtract } 2x \text{ from both sides.} \\ 3-7x &\leq 17 && \text{Simplify both sides.} \end{aligned}$$

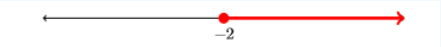
We continue to isolate terms containing x on one side of the inequality. Subtract 3 from both sides.

$$3-7x-3 \leq 17-3 \qquad \text{Subtract 3 from both sides.}$$

$$\begin{aligned} -7x &\leq 14 && \text{Simplify both sides.} \end{aligned}$$

To “undo” multiplying by -7, divide both sides by -7. Because we are dividing both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned} \frac{-7x}{-7} &\geq \frac{14}{-7} && \text{Divide both sides by } -7 \\ x &\geq -2 && \text{Simplify both sides.} \end{aligned}$$



Using set-builder notation, the solution is $\{x : x \geq -2\}$. Using interval notation, the solution is $[-2, \infty)$.

Exercise 2.6.7

Use interval notation to describe the solution of:

$$4 - x > 2x + 1$$

Answer

$(-\infty, 1)$

We clear fractions from an inequality in the usual manner, by multiplying both sides by the least common denominator.

Example 2.6.8

Solve for x : $\quad \frac{3}{4} - \frac{x}{12} > \frac{1}{3}$.

Solution

First, clear the fractions from the inequality by multiplying both sides by the least common denominator, which in this case is 12.

$$\begin{aligned} \frac{3}{4} - \frac{x}{12} &> \frac{1}{3} && \text{Original inequality.} \\ 12\left[\frac{3}{4} - \frac{x}{12}\right] &> \left[\frac{1}{3}\right]12 && \text{Multiply both sides by 12.} \\ 12\left[\frac{3}{4}\right] - 12\left[\frac{x}{12}\right] &> \left[\frac{1}{3}\right]12 && \text{Distribute the 12.} \\ 9 - x &> 4 && \text{Cancel and Multiply.} \end{aligned}$$

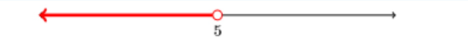
To “undo” adding 9, subtract 9 from both sides.

$$\begin{aligned} 9 - x - 9 &> 4 - 9 && \text{Subtract 9 from both sides.} \\ -x &> -5 && \text{Simplify both sides.} \end{aligned}$$

We could divide both sides by -1, but multiplying both sides by -1 will also do the job. Because we are multiplying both sides by a negative number, we reverse the inequality sign.

$$\begin{aligned} (-1)(-x) &< (-5)(-1) && \text{Multiply both sides by } -1\text{. Reverse the inequality sign.} \\ x &< 5 && \text{Simplify both sides.} \end{aligned}$$

Shade the real numbers less than 5.



Using set-builder notation, the solution is $\{x : x < 5\}$. Using interval notation, the solution is $(-\infty, 5)$.

Exercise 2.6.8

Use interval notation to describe the solution of:

$$\frac{2x}{3}-\frac{3}{4}\geq-\frac{3}{2}$$

Answer

$[-\frac{9}{8},\infty)$

We clear decimals from an inequality in the usual manner, by multiplying both sides by the appropriate power of ten.

Example 2.6.9

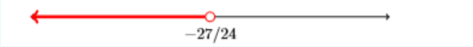
Solve for $x:\quad 3.25-1.2x>4.6$.

Solution

First, clear the decimals from the inequality by multiplying both sides by 100, which moves each decimal point two places to the right.

$$\begin{aligned}
3.25-1.2x &> 4.6 && \text{Original inequality.}\\
325-120x &> 460 && \text{Multiply both sides by 100.}\\
325-120x-325 &> 460-325 && \text{Subtract 325 from both sides.}\\
-120x &> 135 && \text{Simplify both sides.}\\
\frac{-120x}{-120} &< \frac{135}{-120} && \text{Divide both sides by } -120. \text{ Reverse the inequality sign.}\\
x &< -\frac{27}{24} && \text{Reduce to lowest terms.}
\end{aligned}$$

Shade the real numbers less than $-27/24$.



Using set-builder notation, the solution is $\{x : x<-27/24\}$. Using interval notation, the solution is $(-\infty,-27/24)$.

Exercise 2.6.9

Use interval notation to describe the solution of:

$$2.3x-5.62\geq-1.4$$

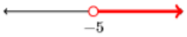
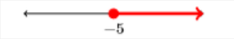
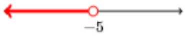
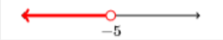
Answer

$\left[\frac{211}{115},\infty\right)$

Summary Table of Set-Builder and Interval Notation

A summary table of the set-builder and interval notation is presented in Table $2.6.1$.

Table $2.6.1$: Examples of set-builder and interval notation.

Shading on the real line	Set-builder	Interval
	$\{x : x>-5\}$	$(-5,\infty)$
	$\{x : x\geq-5\}$	$[-5,\infty)$
	$\{x : x<-5\}$	$(-\infty,-5)$
	$\{x : x\leq-5\}$	$(-\infty,-5]$

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2.E: Solving Linear Equations and Inequalities (Exercises)

2.1: Solving Equations - One Step

In Exercises 1-6, which of the numbers following the given equation are solutions of the given equation? Support your response with work similar to that shown in Example 2.1.1.

1) $x+2=4; \quad 3,2,9,5$

Answer

2

2) $x+6=9; \quad 4,6,3,10$

3) $x-9=4; \quad 16,14,20,13$

Answer

13

4) $x-3=5; \quad 9,15,8,11$

5) $x-3=6; \quad 9,16,10,12$

Answer

9

6) $x-5=7; \quad 15,13,19,12$

In Exercises 7-12, are the following equations equivalent?

7) $x-1=-7$ and $x=-8$

Answer

No

8) $x-7=-8$ and $x=-15$

9) $x-5=-5$ and $x=0$

Answer

Yes

10) $x-3=-3$ and $x=0$

11) $x^2=1$ and $x=1$

Answer

No

12) $x^2=16$ and $x=4$

In Exercises 13-32, solve the given equation for x.

13) $x-20=9$

Answer

29

14) $x-10=5$

15) $16=x-3$

Answer

19

16) $16=x-8$

17) $x+11=20$

Answer

9

18) $x+10=18$

19) $9=x-19$

Answer

28

20) $4=x-11$

21) $20=9+x$

Answer

11

22) $18=7+x$

23) $18=17+x$

Answer

1

24) $15=7+x$

25) $7+x=19$

Answer

12

26) $16+x=17$

27) $x-9=7$

Answer

16

28) $x-2=8$

29) $x+15=19$

Answer

4

30) $x+6=10$

31) $10+x=15$

Answer

5

32) $18+x=19$

In Exercises 33-40, solve the equation and simplify your answer.

33) $x-\frac{4}{9}=\frac{2}{7}$

Answer

$\frac{46}{63}$

34) $x-\frac{1}{3}=\frac{1}{2}$

35) $x+\frac{7}{4}=-\frac{4}{9}$

Answer

$-\frac{79}{36}$

36) $x+\frac{1}{2}=-\frac{9}{7}$

37) $x+\frac{5}{9}=\frac{7}{2}$

Answer

$\frac{53}{18}$

38) $x+\frac{1}{4}=\frac{1}{3}$

39) $x-\frac{9}{8}=-\frac{1}{2}$

Answer

$\frac{5}{8}$

40) $x-\frac{5}{9}=-\frac{2}{3}$

In Exercises 41-46, solve the given equation for x.

41) $-5.1x=-12.75$

Answer

2.5

42) $-3.5x=-22.4$

43) $-6.9x=-58.65$

Answer

8.5

44) $-1.4x=-4.34$

45) $-3.6x=-24.12$

Answer

6.7

46) $-6.4x=-39.68$

In Exercises 47-52, solve the given equation for x.

47) $\frac{x}{2}=-11$

Answer

-22

48) $\frac{x}{8}=12$

49) $\frac{x}{8}=-18$

Answer

-144

50) $\frac{x}{-4}=-17$

51) $\frac{x}{-7}=15$

Answer

-105

52) $\frac{x}{8}=-7$

2.2: Solving Equations - Multiple Steps

In Exercises 1-16, solve the given equation for x.

1) $2x-20=-12$

Answer

4

2) $-4x-1=3$

3) $-11+3x=-44$

Answer

-11

4) $-8+14x=-22$

5) $-5x+17=112$

Answer

-19

6) $3x+12=51$

7) $-16x-14=2$

Answer

-1

8) $4x-4=64$

9) $5-13x=70$

Answer

-5

10) $11+10x=81$

11) $11+10x=81$

Answer

8

12) $-4-16x=-100$

13) $7-x=-7$

Answer

14

14) $20-3x=35$

15) $-4x+14=74$

Answer

-15

16) $-4x+15=27$

In Exercises 17-24, solve the equation and simplify your answer.

17) $\frac{x}{7}-\frac{1}{3}=-\frac{9}{8}$

Answer

$-\frac{133}{24}$

18) $\frac{x}{8}-\frac{8}{3}=-\frac{4}{7}$

19) $\frac{x}{7}+\frac{4}{9}=\frac{3}{2}$

Answer

$\frac{133}{18}$

20) $\frac{x}{5}+\frac{1}{6}=\frac{8}{7}$

21) $\frac{x}{2}+\frac{2}{3}=\frac{4}{7}$

Answer

$-\frac{4}{21}$

22) $\frac{x}{7}+\frac{4}{5}=\frac{3}{4}$

23) $\frac{x}{5}-\frac{9}{2}=-\frac{5}{3}$

Answer

$\frac{85}{6}$

24) $\frac{x}{5}-\frac{8}{9}=-\frac{3}{2}$

In Exercises 26-32, solve each equation.

25) $0.3x + 1.7 = 3.05$

Answer

4.5

26) $-7.2x + 2.9 = 64.10$

27) $1.2x + 5.2 = 14.92$

Answer

8.1

28) $-7.3x + 1.8 = -45.65$

29) $3.5x - 3.7 = -26.10$

Answer

-6.4

30) $-1.4x - 4.7 = 5.80$

31) $-4.7x - 7.4 = -48.29$

Answer

8.7

32) $-5.2x - 7.2 = 38.04$

In Exercises 33-44, solve each equation.

33) $13 - 9x = 11 - 5x$

Answer

$\frac{1}{2}$

34) $11 - 10x = 13 - 4x$

35) $11x + 10 = 19x + 20$

Answer

$-\frac{5}{4}$

36) $20x + 19 = 10x + 13$

37) $11 - 15x = 13 - 19x$

Answer

$\frac{1}{2}$

38) $13 - 11x = 17 - 5x$

39) $9x + 8 = 4 - 19x$

Answer

$-\frac{1}{7}$

40) $10x+8=6-2x$

41) $7x+11=16-18x$

Answer

$\frac{1}{5}$

42) $11x+8=2-17x$

43) $12x+9=4x+7$

Answer

$-\frac{1}{4}$

44) $6x+3=16x+11$

In Exercises 45-56, solve each equation.

45) $8(5x-3)-3(4x+6)=4$

Answer

$\frac{23}{14}$

46) $6(3x-8)-6(4x+6)=3$

47) $2x-4(4-9x)=4(7x+8)$

Answer

$\frac{24}{5}$

48) $4x-9(6-2x)=2(5x+7)$

49) $2(6-2x)-(4x-9)=9$

Answer

$\frac{3}{2}$

50) $2(8-5x)-(2x-6)=4$

51) $3(5x-6)-7(7x+9)=3$

Answer

$-\frac{42}{17}$

52) $9(3x-7)-9(2x+9)=6$

53) $2x-2(4-9x)=8(6x+2)$

Answer

$-\frac{6}{7}$

54) $3x-3(5-9x)=6(8x+2)$

55) $2(7-9x)-(2x-8)=7$

Answer

$\frac{3}{4}$

56) $8(5-2x)-(8x-9)=4$

2.3: Clearing Fractions and Decimals

In Exercises 1-6, simplify the expression.

1) $16\left(\frac{9}{2}x\right)$

Answer

$72x$

2) $27\left(\frac{7}{9}x\right)$

3) $14\left(\frac{3}{2}x\right)$

Answer

$21x$

4) $-12\left(\frac{9}{4}x\right)$

5) $70\left(\frac{9}{7}x\right)$

Answer

$90x$

6) $-27\left(\frac{5}{3}x\right)$

In Exercises 7-18, for each of the following equations, clear fractions by multiplying both sides by the least common denominator. Solve the resulting equation for x and reduce your answer to lowest terms.

7) $-\frac{9}{7}x-\frac{1}{3}=\frac{5}{3}$

Answer

$-\frac{14}{9}$

8) $-\frac{1}{2}x-\frac{3}{4}=-\frac{5}{9}$

9) $\frac{7}{3}x+\frac{5}{9}=\frac{2}{3}x-\frac{4}{3}$

Answer

$-\frac{17}{15}$

10) $\frac{2}{3}x-\frac{9}{4}=-\frac{5}{8}x-\frac{4}{3}$

11) $\frac{9}{4}x-\frac{8}{7}=\frac{3}{2}$

Answer

$\frac{74}{63}$

12) $-\frac{7}{3}x-\frac{2}{9}=-\frac{4}{3}$

13) $-\frac{3}{4}x=-\frac{8}{3}$

Answer

$\frac{32}{9}$

14) $-\frac{2}{3}x = \frac{5}{7}$

15) $x + \frac{3}{4} = \frac{6}{5}$

Answer

$\frac{9}{20}$

16) $x - \frac{2}{9} = \frac{1}{4}$

17) $-\frac{1}{3}x - \frac{4}{3} = -\frac{3}{4}x - \frac{8}{5}$

Answer

$-\frac{16}{25}$

18) $-\frac{6}{7}x - \frac{3}{5} = -\frac{9}{7}x - \frac{1}{2}$

In Exercises 19-32, clear decimals from the given equation by multiplying by the appropriate power of ten, then solve the resulting equation for x. Your final answer should be a fraction reduced to lowest terms.

19) $2.39x + 0.71 = -1.98x + 2.29$

Answer

$\frac{158}{437}$

20) $0.12x + 0.52 = -1.47x - 2.12$

21) $0.4x - 1.55 = 2.14$

Answer

$\frac{369}{40}$

22) $0.8x - 2.18 = 1.49$

23) $2.6x - 2.54 = -2.14x$

Answer

$\frac{127}{237}$

24) $-1.4x - 2.98 = 0.55x$

25) $0.7x = -2.3x - 2.8$

Answer

$-\frac{14}{15}$

26) $3.4x = 1.8x + 2.5$

27) $-4.8x - 2.7 = -1.9$

Answer

$-\frac{1}{6}$

28) $-2.4x+2.5=2.3$

29) $1.7x+2.1=-1.6x+2.5$

Answer

$$\frac{4}{33}$$

30) $-1.2x+0.4=-2.7x-1.9$

31) $2.5x+1.9=0.9x$

Answer

$$-\frac{19}{16}$$

32) $4.4x+0.8=2.8x$

2.4: Formulae

In Exercises 1-30, solve the given formulas for the indicated variable.

1) $F=kx$ for x

Answer

$$x=\frac{F}{k}$$

2) $A=\pi r^2$ for π

3) $E=mc^2$ for m

Answer

$$m=\frac{E}{c^2}$$

4) $v=v_0+at$ for v_0

5) $A=\pi r_1 r_2$ for r_2

Answer

$$r_2=\frac{A}{\pi r_1}$$

6) $y=mx+b$ for b

7) $F=ma$ for a

Answer

$$a=\frac{F}{m}$$

8) $V=lwh$ for l

9) $C=2\pi r$ for r

Answer

$$r=\frac{C}{2\pi}$$

10) $F=kx$ for k

11) $y=mx+b$ for x

Answer

$$x = \frac{y-b}{m}$$

12) $I = \frac{V}{R}$ for V

13) $F = qvB$ for v

Answer

$$v = \frac{F}{qB}$$

14) $x + d = e$ for x

15) $V = \frac{1}{3}\pi r^2 h$ for h

Answer

$$h = \frac{3V}{\pi r^2}$$

16) $P = IRT$ for I

17) $I = \frac{V}{R}$ for R

Answer

$$R = \frac{V}{I}$$

18) $F = \frac{9}{5}C + 32$ for C

19) $F = \frac{kqQ}{r^2}$ for q

Answer

$$q = \frac{r^2 F}{kQ}$$

20) $Ax + By = C$ for y

21) $P = 2W + 2L$ for W

Answer

$$W = \frac{P - 2L}{2}$$

22) $A = \frac{1}{2}h\left(b_1 + b_2\right)$ for b_1

23) $A = \frac{1}{2}h\left(b_1 + b_2\right)$ for h

Answer

$$h = \frac{2A}{b_1 + b_2}$$

24) $A = \frac{a+b+c}{3}$ for b

25) $y - y_0 = m\left(x - x_0\right)$ for m

Answer

$$m = \frac{y - y_0}{x - x_0}$$

26) $A=\frac{1}{2}bh$ for b

27) $F=\frac{GMm}{r^2}$ for M

Answer

$$M=\frac{r^2F}{Gm}$$

28) $A=\frac{a+b+c}{3}$ for a

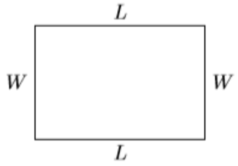
29) $d=vt$ for v

Answer

$$v=\frac{d}{t}$$

30) $x+d=e$ for d

31) Let W and L represent the width and length of a rectangle, respectively, and let A represent its area.



The area of the rectangle is given by the formula

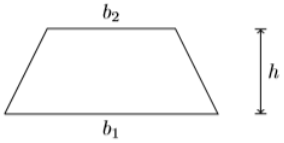
$$A=LW$$

Solve this formula for L. Then, given that the area of the rectangle is $A=1073$ square meters and its width is $W=29$ meters, determine its length.

Answer

37 square meters

32) Let b_1 and b_2 represent the parallel bases of a trapezoid and let h represent its height.

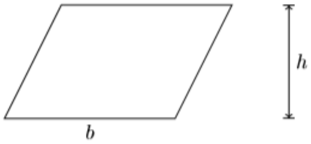


The area of the trapezoid is given by the formula:

$$A=\frac{1}{2}(b_1+b_2)h$$

Solve this formula for b_1. Then, given that the area is $A=2457$ square centimeters, the second base is $b_2=68$ centimeters, and the height is $h=54$ centimeters, find the length b_1 of the first base.

33) A parallelogram is a quadrilateral (four sided figure) whose opposite sides are parallel.



The area of the parallelogram is computed using the formula:

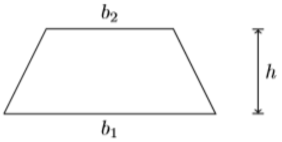
$$A = bh$$

Solve this formula for b. Next, given that the area is $A = 2418$ square feet and the height is $h = 31$ feet, find the length of the base of the parallelogram.

Answer

78 feet

34) Let b_1 and b_2 represent the parallel bases of a trapezoid and let h represent its height.

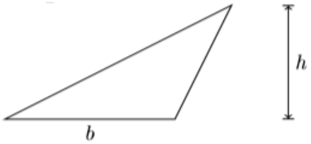


The area of the trapezoid is given by the formula:

$$A = \frac{1}{2}(b_1 + b_2)h$$

Solve this formula for h. Then, given that the area is $A = 3164$ square yards, the bases are $b_1 = 38$ yards and $b_2 = 75$ yards, find the height h of the trapezoid.

35) Let b and h represent the length of the base and the height of a triangle, respectively, and let A represent the area of the triangle.



The area of the triangle is computed using the formula:

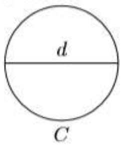
$$A = \frac{1}{2}bh$$

Solve this formula for b. Next, given that the area is $A = 1332$ square inches and the height is $h = 36$ inches, find the length of the base of the triangle.

Answer

74 inches

36) The circumference of a circle, somewhat like the term perimeter, is the distance around the circle. The diameter of a circle is a line segment drawn through the center of the circle.

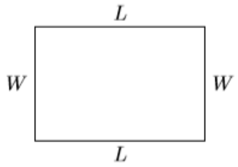


Since the time of the ancient Greeks, it has been known that the ratio of the circumference to the diameter is a constant, denoted by the symbol π.

$$\frac{C}{d} = \pi$$

Solve the formula for d. Then, given $C = 188.4$ yards and $\pi = 3.14$, find the length of the diameter d.

37) Let W and L represent the width and length of a rectangle, respectively, and let P represent its perimeter.



The perimeter (distance around) of the rectangle is found by summing its four sides, then combining like terms.

$$P = 2W + 2L$$

Solve $P = 2W + 2L$ for W. Then, given that the perimeter is $P = 256$ meters and the length is $L = 73$ meters, use your result to calculate the width.

Answer

55 meters

2.5: Applications

1) Two angles are said to be complementary if their sum is ninety degrees. Suppose you have two complementary angles such that the second angle is 6 degrees larger than 2 times the measure of the first angle. Find the angles.

Answer

28, 62 degrees

2) Two angles are said to be complementary if their sum is ninety degrees. Suppose you have two complementary angles such that the second angle is 10 degrees larger than 3 times the measure of the first angle. Find the angles.

3) Two angles are said to be supplementary if their sum is 180 degrees. Suppose you have two supplementary angles such that the second angle is 10 degrees larger than 4 times the measure of the first angle. Find the angles.

Answer

34, 146 degrees

4) Two angles are said to be supplementary if their sum is 180 degrees. Suppose you have two supplementary angles such that the second angle is 12 degrees larger than 5 times the measure of the first angle. Find the angles.

5) The three sides of a triangle are consecutive integers. If the perimeter (sum of the three sides) of the triangle is 483 meters, find the length of each side of the triangle.

Answer

160, 161, 162 meters

6) The three sides of a triangle are consecutive integers. If the perimeter (sum of the three sides) of the triangle is 486 yards, find the length of each side of the triangle.

7) Four less than eight times a certain number is -660. Find the number.

Answer

-82

8) Nine less than five times a certain number is 141. Find the number.

9) Alan is hiking a trail that is 70 miles long. After several days, he is four times as far from the beginning of the trail as he is from the end. How much further does he have to hike?

Answer

14 miles

10) Joe is hiking a trail that is 30 miles long. After several days, he is two times as far from the beginning of the trail as he is from the end. How much further does he have to hike?

11) Martha takes roll in her sixth grade grade class and finds that 2 students are missing. If her actual class size is 36 students, what percentage of her class is absent? Round your answer to the nearest percent.

Answer

6%

12) Alice takes roll in her first grade grade class and finds that 7 students are missing. If her actual class size is 37 students, what percentage of her class is absent? Round your answer to the nearest percent.

13) Lily cuts a piece of yarn into three pieces. The second pieces is 3 times as long as the first piece, and the third piece is 6 centimeters longer than the first piece. If the total length of the yarn is 211 centimeters, find the lengths of each of the three pieces.

Answer

41, 123, and 47 centimeters

14) Jane cuts a piece of twine into three pieces. The second pieces is 7 times as long as the first piece, and the third piece is 5 feet longer than the first piece. If the total length of the twine is 320 feet, find the lengths of each of the three pieces.

15) The three sides of a triangle are consecutive even integers. If the perimeter (sum of the three sides) of the triangle is 450 yards, find the length of each side of the triangle.

Answer

$148, 150, 152$ yards

16) The three sides of a triangle are consecutive even integers. If the perimeter (sum of the three sides) of the triangle is 318 feet, find the length of each side of the triangle.

17) The perimeter of a triangle is 414 yards. The second side of the triangle is 7 times as long as the first side and the third side of the triangle is 9 yards longer than the first side. Find the lengths of each of the three sides of the triangle.

Answer

45, 315, and 54 yards

18) The perimeter of a triangle is 54 inches. The second side of the triangle is 2 times as long as the first side and the third side of the triangle is 6 inches longer than the first side. Find the lengths of each of the three sides of the triangle.

19) The sum of three consecutive odd integers is -543. Find the smallest of the three consecutive odd integers.

Answer

-183

20) The sum of three consecutive odd integers is -225. Find the smallest of the three consecutive odd integers.

21) The sum of the angles of a triangle is 180°. In triangle $\triangle ABC$, the degree measure of angle B is 4 times the degree measure of angle A. The degree measure of angle C is 30 degrees larger than the degree measure of angle A. Find the degree measures of each angle of triangle $\triangle ABC$.

Answer

$25^{\circ}, 100^{\circ}, 55^{\circ}$

22) The sum of the angles of a triangle is 180°. In triangle $\triangle ABC$, the degree measure of angle B is 4 times the degree measure of angle A. The degree measure of angle C is 60 degrees larger than the degree measure of angle A. Find the degree measures of each angle of triangle $\triangle ABC$.

23) The sum of three consecutive integers is -384. Find the largest of the three consecutive integers.

Answer

\(−127\

24) The sum of three consecutive integers is -501. Find the largest of the three consecutive integers.

25) Seven more than two times a certain number is 181. Find the number.

Answer

87

26) Nine more than two times a certain number is 137. Find the number.

27) The three sides of a triangle are consecutive odd integers. If the perimeter (sum of the three sides) of the triangle is 537 feet, find the length of each side of the triangle.

Answer

177, 179, 181 feet

28) The three sides of a triangle are consecutive odd integers. If the perimeter (sum of the three sides) of the triangle is 471 centimeters, find the length of each side of the triangle.

29) A store advertises that it is offering a 14% discount on all articles purchased at the store. If Yao pays $\$670.80$ for an article, what was the marked price for the article?

Answer

$\$780.00$

30) A store advertises that it is offering a 12% discount on all articles purchased at the store. If Roberto pays $\$560.56$ for an article, what was the marked price for the article?

31) The sum of three consecutive even integers is -486. Find the smallest of the three consecutive even integers.

Answer

-164

32) The sum of three consecutive even integers is -354. Find the smallest of the three consecutive even integers.

33) Burt inherits $\$45,500$. He decides to invest part of the inheritance in a mutual fund and the remaining part in a certificate of deposit. If the amount invested in the certificate of deposit is $\$3,500$ more than 6 times the amount invested in the mutual fund, find the amount invested in each account.

Answer

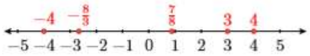
$\$6,000, \$39,500$

34) Phoenix inherits $\$12,000$. He decides to invest part of the inheritance in a mutual fund and the remaining part in a certificate of deposit. If the amount invested in the certificate of deposit is $\$3,000$ more than 8 times the amount invested in the mutual fund, find the amount invested in each account.

2.6: Inequalities

1) Draw a number line, then plot the numbers $4, 3, -4, 7/8$, and $-8/3$ on your number line. Label each point with its value. Finally, list the numbers in order, from smallest to largest.

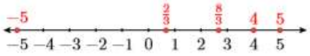
Answer



2) Draw a number line, then plot the numbers $5, 3, -4, 5/7$, and $-4/3$ on your number line. Label each point with its value. Finally, list the numbers in order, from smallest to largest.

3) Draw a number line, then plot the numbers $-5, 5, 4, 2/3$, and $8/3$ on your number line. Label each point with its value. Finally, list the numbers in order, from smallest to largest.

Answer

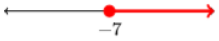


4) Draw a number line, then plot the numbers $-3, -2, 4, 1/3$, and $5/2$ on your number line. Label each point with its value. Finally, list the numbers in order, from smallest to largest.

In Exercises 5-20, shade each of the following sets on a number line.

5) $\{x : x \geq -7\}$

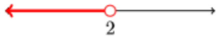
Answer



6) $\{x : x \geq -1\}$

7) $\{x : x < 2\}$

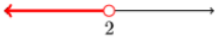
Answer



8) $\{x : x < -6\}$

9) $(-\infty, 2)$

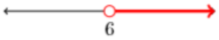
Answer



10) $(-\infty, -9)$

11) $(6, \infty)$

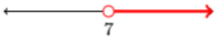
Answer



12) $(5, \infty)$

13) $\{x : x > 7\}$

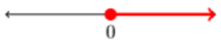
Answer



14) $\{x : x > -8\}$

15) $[0, \infty)$

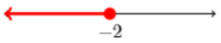
Answer



16) $[7, \infty)$

17) $\{x : x \leq -2\}$

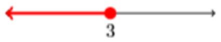
Answer



18) $\{x : x \leq 7\}$

19) $(-\infty, 3]$

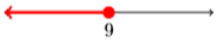
Answer



20) $(-\infty, -1]$

In Exercises 21-28, use set-builder notation to describe the shaded region on the given number line.

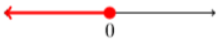
21)



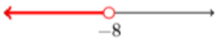
Answer

$\{x : x \leq 9\}$

22)



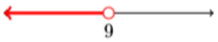
23)



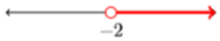
Answer

$\{x : x < -8\}$

24)



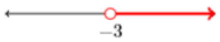
25)



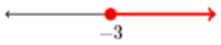
Answer

$\{x : x > -2\}$

26)



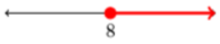
27)



Answer

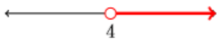
$\{x : x \geq -3\}$

28)



In Exercises 29-36, use interval notation to describe the shaded region on the given number line.

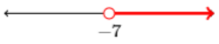
29)



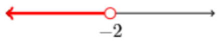
Answer

$(4, \infty)$

30)



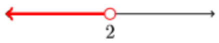
31)



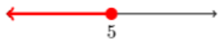
Answer

$(-\infty, -2)$

32)



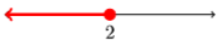
33)



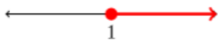
Answer

$(-\infty, 5]$

34)



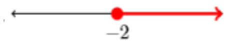
35)



Answer

$[1, \infty)$

36)



In Exercises 37-44, solve each of the given inequalities. Sketch the solution on a number line, then use set-builder and interval notation to describe your solution.

37) $x+10<19$

Answer

$(-\infty, 9)$

38) $x+17 \geq 7$

39) $4x<8$

Answer

$(-\infty, 2)$

40) $16x \geq -2$

41) $-2x \leq -2$

Answer

$[1, \infty)$

42) $-18x>-20$

43) $x-18>-10$

Answer

$(8, \infty)$

44) $x-8 \leq -18$

In Exercises 45-62, solve each of the given inequalities. Sketch the solution on a number line, then use set-builder and interval notation to describe your solution.

45) $-5x-6 \geq 4-9x$

Answer

$[5/2, \infty)$

46) $2x-7 \geq -3-4x$

47) $16x-6 \leq 18$

Answer

$(-\infty, 3/2]$

48) $8x-14 \leq -12$

49) $-14x-6 \geq -10-4x$

Answer

$(-\infty, 2/5]$

50) $-13x-4 \geq -2-5x$

51) $5x+18<38$

Answer

$(-\infty, 4)$

52) $9x + 16 < 79$

53) $-16x - 5 \geq -11 - 6x$

Answer

$(-\infty, 3/5]$

54) $-11x - 7 \geq -15 - 5x$

55) $2x - 9 \geq 5 - 8x$

Answer

$[7/5, \infty)$

56) $-3x - 6 \geq -2 - 9x$

57) $-10x - 4 \leq 18$

Answer

$[-11/5, \infty)$

58) $-6x - 14 \leq 1$

59) $-12x + 4 < -56$

Answer

$(5, \infty)$

60) $-18x + 6 < -12$

61) $15x + 5 < 6x + 2$

Answer

$(-\infty, -1/3)$

62) $12x + 8 < 3x + 5$

In Exercises 63-76, solve each of the given inequalities. Sketch the solution on a number line, then use set-builder and interval notation describe your solution.

63) $\frac{3}{2}x > \frac{9}{8}$

Answer

$(3/4, \infty)$

64) $\frac{6}{7}x > \frac{3}{4}$

65) $x + \frac{3}{2} < \frac{9}{5}$

Answer

$(-\infty, 3/10)$

66) $x + \frac{1}{4} < -\frac{1}{5}$

67) $\frac{4}{7} - \frac{1}{6}x \leq \frac{4}{3}x - \frac{1}{2}$

Answer

$[5/7, \infty)$

68) $\frac{5}{3} - \frac{3}{4}x \leq \frac{7}{4}x - \frac{3}{5}$

69) $x - \frac{3}{8} \geq -\frac{9}{7}$

Answer

$[-51/56, \infty)$

70) $x - \frac{7}{2} \geq \frac{1}{5}$

71) $\frac{6}{5}x \leq -\frac{4}{7}$

Answer

$[10/21, \infty)$

72) $\frac{4}{3}x \leq \frac{2}{9}$

73) $-\frac{6}{5}x - \frac{7}{3} \leq \frac{5}{9} - \frac{2}{9}x$

Answer

$[-65/22, \infty)$

74) $-\frac{3}{7}x - \frac{1}{2} \leq \frac{3}{2} - \frac{2}{7}x$

75) $\frac{9}{7}x + \frac{9}{2} > \frac{1}{7}x + \frac{7}{2}$

Answer

$(-7/8, \infty)$

76) $\frac{5}{7}x + \frac{9}{2} > \frac{1}{3}x + \frac{5}{2}$

In Exercises 77-84, solve each of the given inequalities. Sketch the solution on a number line, then use set-builder and interval notation containing fractions in reduced form to describe your solution.

77) $-3.7x - 1.98 \leq 3.2$

Answer

$[-7/5, \infty)$

78) $-3.6x - 3.32 \leq 0.8$

79) $-3.4x + 3.5 \geq 0.9 - 2.2x$

Answer

$(-\infty, 13/6]$

80) $-2.6x + 3.1 \geq -2.9 - 1.7x$

81) $-1.3x + 2.9 > -2.6 - 3.3x$

Answer

$(-11/4, \infty)$

82) $2.5x + 2.1 > 1.4 - 3.8x$

83) $2.2x + 1.9 < -2.3$

Answer

$(-\infty, -21/11)$

84) $1.6x + 1.2 < 1.6$

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CHAPTER OVERVIEW

3: Introduction to Graphing

3.1: Graphing Equations by Hand
3.2: The Graphing Calculator
3.3: Rates and Slope
3.4: Slope-Intercept Form of a Line
3.5: Point-Slope Form of a Line
3.6: Standard Form of a Line
3.E: Introduction to Graphing (Exercises)

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3.1: Graphing Equations by Hand

We begin with the definition of an *ordered pair*.

Ordered Pair

The construct (x, y), where x and y are any real numbers, is called an ordered pair of real numbers.

$(4,3)$, $(-3,4)$, $(-2,-3)$, and $(3,-1)$ are examples of ordered pairs.

Order Matters

Pay particular attention to the phrase “ordered pairs.” Order matters. Consequently, the ordered pair (x, y) is not the same as the ordered pair (y, x), because the numbers are presented in a different order.

The Cartesian Coordinate System

Pictured in Figure $3.1.1$ is a Cartesian Coordinate System. On a grid, we’ve created two real lines, one horizontal labeled x (we’ll refer to this one as the x-axis), and the other vertical labeled y (we’ll refer to this one as the y-axis).

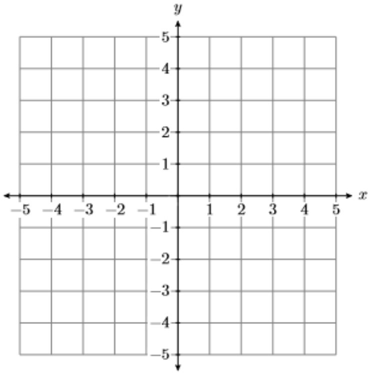


Figure $3.1.1$: The Cartesian Coordinate System.

Two Important Points:

Here are two important points to be made about the horizontal and vertical axes in Figure $3.1.1$.

1. As you move from left to right along the horizontal axis (the x-axis in Figure $3.1.1$), the numbers grow larger. The positive direction is to the right, the negative direction is to the left.
2. As you move from bottom to top along the vertical axis (the y-axis in Figure $3.1.1$), the numbers grow larger. The positive direction is upward, the negative direction is downward.

Additional Comments:

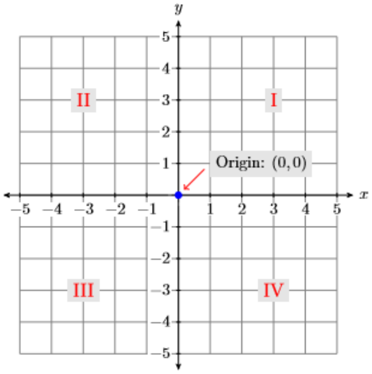


Figure $3.1.2$: Numbering the quadrants and indicating the coordinates of the origin.

Two additional comments are in order:

3. The point where the horizontal and vertical axes intersect in Figure $3.1.2$ is called the origin of the coordinate system. The origin has coordinates $(0,0)$.
4. The horizontal and vertical axes divide the plane into four quadrants, numbered $\mathrm{I}, \mathrm{II}, \mathrm{MI},$ and IV (roman numerals for one, two, three, and four), as shown in Figure $3.1.2$. Note that the quadrants are numbered in a counter-clockwise order.

Note

Rene Descartes (1596-1650) was a French philosopher and mathematician who is well known for the famous phrase“cogito ergo sum” (I think, therefore I am), which appears in his Discours de la methode pour bien conduire sa raison, et chercher la verite dans les sciences (Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences). In that same treatise, Descartes introduces his coordinate system, a method for representing points in the plane via pairs of real numbers. Indeed, the Cartesian plane of modern day is so named in honor of Rene Descartes, who some call the “Father of Modern Mathematics”

Plotting Ordered Pairs

Before we can plot any points or draw any graphs, we first need to set up a Cartesian Coordinate System on a sheet of graph paper? How do we do this? What is required?

how to setup cartesian coordinate system

Draw and label each axis.

If we are going to plot points (x,y), then, on a sheet of graph paper, perform each of the following initial tasks.

1. Use a ruler to draw the horizontal and vertical axes.
2. Label the horizontal axis as the x-axis and the vertical axis as the y-axis.

We don’t always label the horizontal axis as the x-axis and the vertical axis as the y-axis. For example, if we want to plot the velocity of an object as a function of time, then we would be plotting points (t, v). In that case, we would label the horizontal axis as the t-axis and the vertical axis as the v-axis.

Indicate the scale on each axis.

3. Label at least one vertical gridline with its numerical value.
4. Label at least one horizontal gridline with its numerical value.

The scales on the horizontal and vertical axes may differ. However, on each axis, the scale must remain consistent. That is, as you count to the right from the origin on the x-axis, if each gridline represents one unit, then as you count to the left from the origin on the x-axis, each gridline must also represent one unit. Similar comments are in order for the y-axis, where the scale must also be consistent, whether you are counting up or down.

The result of this first step is shown in Figure $3.1.3$.

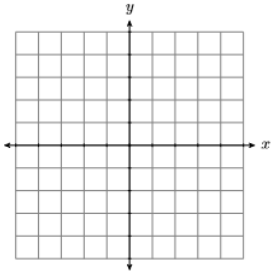


Figure 3.1.3: Draw and label each axis.

An example is shown in Figure 3.1.4. Note that the scale indicated on the x-axis indicates that each gridline counts as 1-unit as we count from left-to-right. The scale on the y-axis indicates that each gridlines counts as 2-units as we count from bottom-to-top.

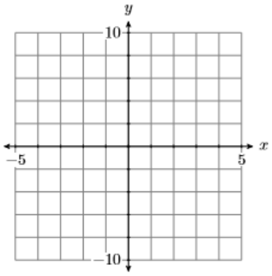


Figure 3.1.4: Indicate the scale on each axis.

Now that we know how to set up a Cartesian Coordinate System on a sheet of graph paper, here are two examples of how we plot points on our coordinate system.

To plot the ordered pair $(4, 3)$, start at the origin and move 4 units to the right along the horizontal axis, then 3 units upward in the direction of the vertical axis.

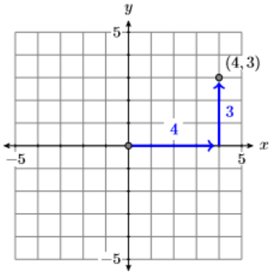


Figure 3.1.5: Plot of $(4, 3)$.

To plot the ordered pair $(-2, -3)$, start at the origin and move 2 units to the left along the horizontal axis, then 3 units downward in the direction of the vertical axis.

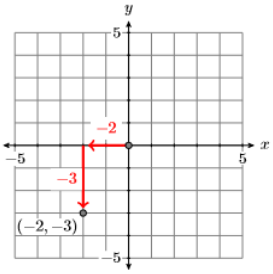


Figure 3.1.6: Plot of $(-2, -3)$.

Continuing in this manner, each ordered pair (x, y) of real numbers is associated with a unique point in the Cartesian plane. Vice-versa, each point in the Cartesian point is associated with a unique ordered pair of real numbers. Because of this association, we begin to use the words "point" and "ordered pair" as equivalent expressions, sometimes referring to the "point" (x, y) and other times to the "ordered pair" (x, y).

Example $3.1.1$

Identify the coordinates of the point P in Figure $3.1.7$.

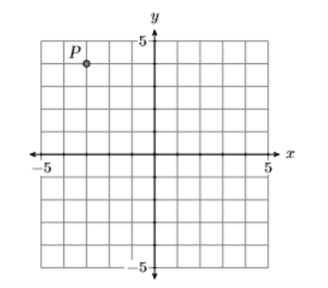


Figure $3.1.7$: Identify the coordinates of the point P.

Solution

In Figure $3.1.8$, start at the origin, move 3 units to the left and 4 units up to reach the point P. This indicates that the coordinates of the point P are $(-3, 4)$.

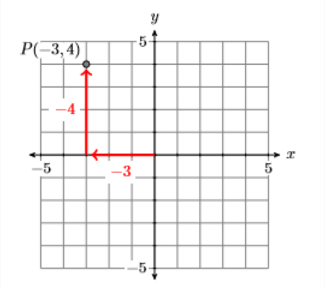
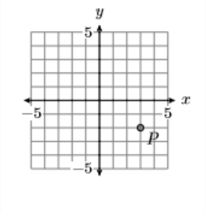


Figure $3.1.8$: Start at the origin, move 3 units left and 4 units up.

Exercise $3.1.1$

Identify the coordinates of the point P in the graph below.



Answer

$(3, -2)$

Equations in Two Variables

Note

The variables do not have to always be x and y. For example, the equation $v = 2 + 3.2t$ is an equation in two variables, v and t.

The equation $y=x+1$ is an equation in two variables, in this case x and y. Consider the point $(x,y)=(2,3)$. If we substitute 2 for x and 3 for y in the equation $y=x+1$, we get the following result:

$$\begin{aligned}&y=x+1 \quad \text{Original equation.}\\&3=2+1 \quad \text{Substitute: } 2 \text{ for } x, 3 \text{ for } y\\&3=3 \quad \text{Simplify both sides.}\end{aligned}$$

Because the last line is a true statement, we say that $(2,3)$ is a solution of the equation $y=x+1$. Alternately, we say that $(2,3)$ satisfies the equation $y=x+1$. On the other hand, consider the point $(x,y)=(-3,1)$. If we substitute -3 for x and 1 for y in the equation $y=x+1$, we get the following result.

$$\begin{aligned}&y=x+1 \quad \text{Original equation.}\\&1=-3+1 \quad \text{Substitute: } -3 \text{ for } x, 1 \text{ for } y\\&1=-2 \quad \text{Simplify both sides.}\end{aligned}$$

Because the last line is a false statement, the point $(-3,1)$ is **not** a solution of the equation $y=x+1$; that is, the point $(-3,1)$ does **not** satisfy the equation $y=x+1$.

Solutions of an equation in two variables

Given an equation in the variables x and y and a point $(x,y)=(a,b)$, if upon subsituting a for x and b for y a true statement results, then the point $(x,y)=(a,b)$ is said to be a solution of the given equation. Alternately, we say that the point $(x,y)=(a,b)$ satisfies the given equation.

Example 3.1.2

Which of the ordered pairs $(0,-3)$ and $(1,1)$ satisfy the equation $y=3x-2$?

Solution

Substituting the ordered pairs $(0,-3)$ and $(1,1)$ into the equation $y=3x-2$ lead to the following results:

Consider $(x,y)=(0,-3)$. Substitute 0 for x and -3 for y:

$$\begin{aligned}y&=3x-2\\-3&=3(0)-2\\-3&=-2\end{aligned}$$

The resulting statement is false.

Consider $(x,y)=(1,1)$. Substitute 1 for x and 1 for y:

$$\begin{aligned}y&=3x-2\\1&=3(1)-2\\1&=1\end{aligned}$$

The resulting statement is true.

Thus, the ordered pair $(0,-3)$ does not satisfy the equation $y=3x-2$, but the ordered pair $(1,1)$ does satisfy the equation $y=3x-2$.

Exercise 3.1.2

Which of the ordered pairs $(-1,3)$ and $(2,1)$ satisfy the equation $y=2x+5$?

Answer

$(-1,3)$

Graphing Equations in Two Variables

Let's first define what is meant by the graph of an equation in two variables.

The graph of an equation

The graph of an equation is the set of all points that satisfy the given equation.

Example 3.1.3

Sketch the graph of the equation $y=x+1$.

Solution

The definition requires that we plot all points in the Cartesian Coordinate System that satisfy the equation $y=x+1$. Let's first create a table of points that satisfy the equation. Start by creating three columns with headers x, y, and (x, y), then select some values for x and put them in the first column.

Take the first value of x, namely $x=-3$, and substitute it into the equation $y=x+1$.

$$\begin{aligned} &y=x+1\\ &y=-3+1\\ &y=-2 \end{aligned}$$

x	$y=x+1$	(x, y)
-3	-2	$(-3,-2)$
-2		
-1		
0		
1		
2		
3		

Thus, when $x=-3$, we have $y=-2$. Enter this value into the table.

Continue substituting each tabular value of x into the equation $y=x+1$ and use each result to complete the corresponding entries in the table.

$$\begin{aligned} &y=-3+1=-2\\ &y=-2+1=-1\\ &y=-1+1=0\\ &y=0+1=1\\ &y=1+1=2\\ &y=2+1=3\\ &y=3+1=4 \end{aligned}$$

x	$y=x+1$	(x, y)
-3	-2	$(-3,-2)$
-2	-1	$(-2,-1)$
-1	0	$(-1,0)$
0	1	$(0,1)$
1	2	$(1,2)$
2	3	$(2,3)$
3	4	$(3,4)$

The last column of the table now contains seven points that satisfy the equation $y=x+1$. Plot these points on a Cartesian Coordinate System (see Figure \(\PageIndex{9}\)).

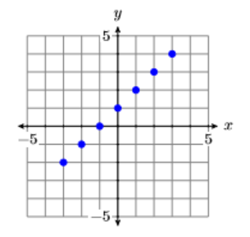


Figure 3.1.9: Seven points that satisfy the equation $y=x+1$.

In Figure 3.1.9, we have plotted seven points that satisfy the given equation $y=x+1$. However, the definition requires that we plot all points that satisfy the equation. It appears that a pattern is developing in Figure 3.1.9, but let's calculate and plot a few more points in order to be sure. Add the x-values -2.5, -1.5, -0.5, 0.5, 1.5, and 2.5 to the x-column of the table, then use the equation $y=x+1$ to evaluate y at each one of these x-values.

$$\begin{aligned}
y &= -2.5+1=-1.5\\
y &= -1.5+1=-0.5\\
y &= -0.5+1=0.5\\
y &= 0.5+1=1.5\\
y &= 1.5+1=2.5\\
y &= 2.5+1=3.5
\end{aligned}$$

x	$y=x+1$	(x,y)
-2.5	-1.5	$(-2.5,-1.5)$
-1.5	-0.5	$(-1.5,-0.5)$
-0.5	0.5	$(-0.5,0.5)$
0.5	1.5	$(0.5,1.5)$
1.5	2.5	$(1.5,2.5)$
2.5	3.5	$(2.5,3.5)$

Add these additional points to the graph in Figure 3.1.9 to produce the image shown in Figure 3.1.10.

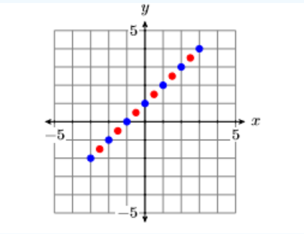


Figure 3.1.10: Adding additional points to the graph of $y=x+1$.

There are an infinite number of points that satisfy the equation $y=x+1$. In Figure 3.1.10, we've plotted only 13 points that satisfy the equation. However, the collection of points plotted in Figure 3.1.10 suggest that if we were to plot the remainder of the points that satisfy the equation $y=x+1$, we would get the graph of the line shown in Figure 3.1.11.

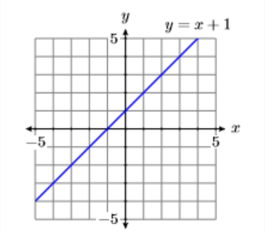
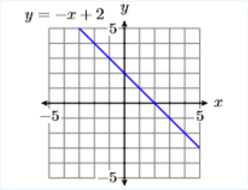


Figure 3.1.11: The graph of $y=x+1$ is a line.

Exercise $3.1.3$

Sketch the graph of the equation $y=-x+2$.

Answer



Guidelines and Requirements

Example $3.1.3$ suggests that we should use the following guidelines when sketching the graph of an equation.

Guidelines for drawing the graph of an equation

When asked to draw the graph of an equation, perform each of the following steps:

1. Set up and calculate a table of points that satisfy the given equation.
2. Set up a Cartesian Coordinate System on graph paper and plot the points in your table on the system. Label each axis (usually x and y) and indicate the scale on each axis.
3. If the number of points plotted are enough to envision what the shape of the final curve will be, then draw the remaining points that satisfy the equation as imagined. Use a ruler if you believe the graph is a line. If the graph appears to be a curve, freehand the graph without the use of a ruler.
4. If the number of plotted points do not provide enough evidence to envision the final shape of the graph, add more points to your table, plot them, and try again to envision the final shape of the graph. If you still cannot predict the eventual shape of the graph, keep adding points to your table and plotting them until you are convinced of the final shape of the graph.

Here are some additional requirements that must be followed when sketching the graph of an equation.

Graph paper, lines, curves, and rulers.

The following are requirements for this class:

5. All graphs are to be drawn on graph paper.
6. All lines are to be drawn with a ruler. This includes the horizontal and vertical axes.
7. If the graph of an equation is a curve instead of a line, then the graph should be drawn freehand, without the aid of a ruler.

Using the TABLE Feature of the Graphing Calculator

As the equations become more complicated, it can become quite tedious to create tables of points that satisfy the equation. Fortunately, the graphing calculator has a TABLE feature that enables you to easily construct tables of points that satisfy the given equation.

Example $3.1.4$

Use the graphing calculator to help create a table of points that satisfy the equation $y=x^2-7$. Plot the points in your table. If you don't feel that there is enough evidence to envision what the final shape of the graph will be, use the calculator to add more points to your table and plot them. Continue this process until your are convinced of the final shape of the graph.

Solution

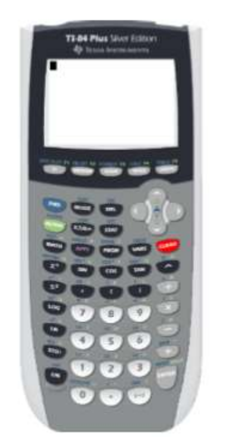


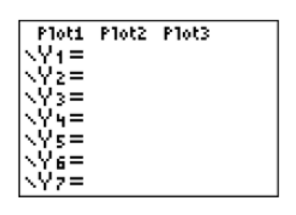
Figure 3.1.12: The graphing calculator.

The first step is to load the equation $y=x^2-7$ into the **Y**= menu of the graphing calculator. The topmost row of buttons on your calculator (see Figure 3.1.12) have the following appearance:

Y= WINDOW ZOOM TRACE GRAPH

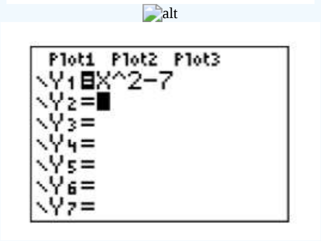
Figure 3.1.13. Use the up-and-down arrow keys (see Figure 3.1.15) to move the cursor after **Y1**= in the **Y**= menu, then use the following keystrokes to enter the equation $y=x^2-7$. The result is shown in Figure 3.1.14.

X,T,θ,n ∧ 2 − 7 ENTER



```
Plot1 Plot2 Plot3
\Y1=
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

alt



```
Plot1 Plot2 Plot3
\Y1=X^2-7
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 3.1.13: Opening the **Y**= menu. ***Figure*** 3.1.14: Entering the equation $y=x^2-7$.

Figure 3.1.15: Arrow keys move the cursor.

The next step is to “set up” the table. First, note that the calculator has symbolism printed on its case above each of its buttons. Above the WINDOW button you’ll note the phrase TBLSET. Note that it is in the same color as the 2ND button. Thus, to open the setup window for the table, enter the following keystrokes.

alt

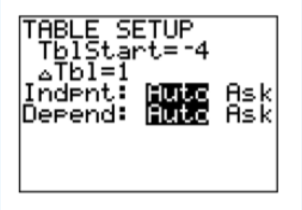
2ND WINDOW

Figure 3.1.16) equal to the first x-value you want to see in your table. In this case, enter -4 after TblStart. SetTbl to the increment you want for your x-values. In this case, set Tbl equal to 1. Finally, set both the independent and dependent variables to “automatic.” In each case, use the arrow keys to highlight the word AUTO and press ENTER. The result is shown in Figure 3.1.16.

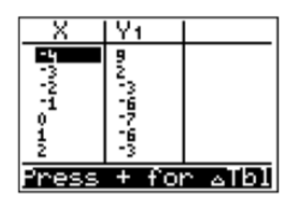
Next, note the word TABLE above the GRAPH button is in the same color as the 2ND key. To open the TABLE, enter the following keystrokes.

2ND GRAPH

Figure 3.1.17. Note that you can use the up-and-down arrow keys to scroll through the table.



```
TABLE SETUP
 TblStart=-4
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

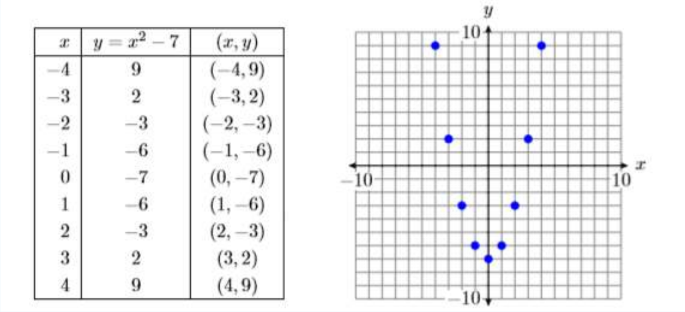


X	Y₁
-4	9
-3	2
-2	-3
-1	-6
0	-7
1	-6
2	-3

Press + for ΔTbl

Figure 3.1.16**:** Opening the TBLSET menu. **Figure** 3.1.17**:** Table of points satisfying $y=x^2-7$.

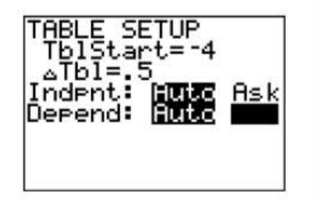
Next, enter the results from your calculator’s table into a table on a sheet of graph paper, then plot the points in the table. The results are shown in Figure 3.1.18.



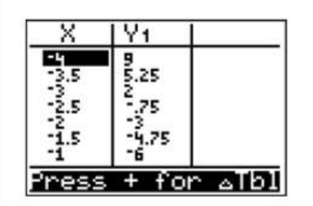
x	$y=x^2-7$	(x,y)
-4	9	$(-4,9)$
-3	2	$(-3,2)$
-2	-3	$(-2,-3)$
-1	-6	$(-1,-6)$
0	-7	$(0,-7)$
1	-6	$(1,-6)$
2	-3	$(2,-3)$
3	2	$(3,2)$
4	9	$(4,9)$

Figure 3.1.18**:** Plotting nine points that satisfy the equation $y=x^2-7$.

In Figure 3.1.18, the eventual shape of the graph of $y=x^2-7$ may be evident already, but let’s add a few more points to our table and plot them. Open the table “setup” window again by pressing 2ND WINDOW. Set TblStart to -4 again, then set the increment Tbl to 0.5. The result is shown in Figure 3.1.19.



```
TABLE SETUP
 TblStart=-4
 ΔTbl=.5
Indpnt: Auto Ask
Depend: Auto
```

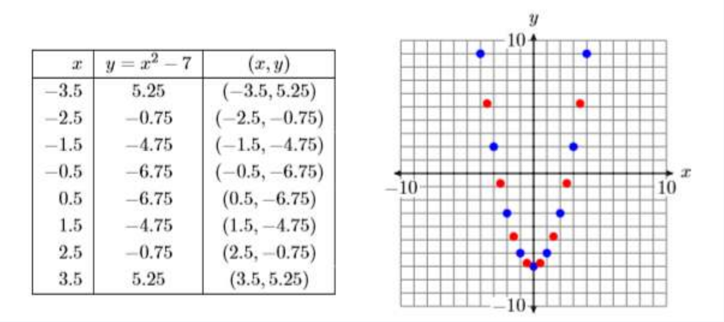


X	Y1
-4	9
-3.5	5.25
-3	2
-2.5	-.75
-2	-3
-1.5	-4.75
-1	-6

Press + for ΔTbl

Figure 3.1.19: SetTbl equal to 0.5. **Figure** 3.1.20: More points satisfying $y = x^2 - 7$.

Add these new points to the table on your graph paper and plot them (see Figure 3.1.21).



x	$y = x^2 - 7$	(x, y)
-3.5	5.25	$(-3.5, 5.25)$
-2.5	-0.75	$(-2.5, -0.75)$
-1.5	-4.75	$(-1.5, -4.75)$
-0.5	-6.75	$(-0.5, -6.75)$
0.5	-6.75	$(0.5, -6.75)$
1.5	-4.75	$(1.5, -4.75)$
2.5	-0.75	$(2.5, -0.75)$
3.5	5.25	$(3.5, 5.25)$

Figure 3.1.21: Adding additional points to the graph of $y = x^2 - 7$.

There are an infinite number of points that satisfy the equation $y = x^2 - 7$. In Figure 3.1.21, we've plotted only 17 points that satisfy the equation $y = x^2 - 7$. However, the collection of points in Figure 3.1.21 suggest that if we were to plot the remainder of the points that satisfy the equation $y = x^2 - 7$, the result would be the curve (called a parabola) shown in Figure 3.1.22.

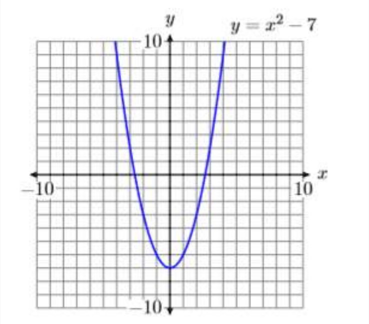


Figure 3.1.22: The graph of $y = x^2 - 7$ is a curve called a parabola.

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3.2: The Graphing Calculator

It’s time to learn how to use a graphing calculator to sketch the graph of an equation. We will use the TI-84 graphing calculator in this section, but the skills we introduce will work equally well on the ancient TI-82 and the less ancient TI-83 graphing calculators.

Figure 3.2.1: The Graphing Calculator.

In this introduction to the graphing calculator, you will need to use several keys on the upper half of the graphing calculator (see Figure 3.2.2). The upand-down and left-and-right arrow keys are located in the upper right corner of Figure 3.2.2. These are used for moving the cursor on the calculator view screen and various menus. Immediately below these arrow keys is the CLEAR button, used to clear the view screen and equations in the **Y**= menu.

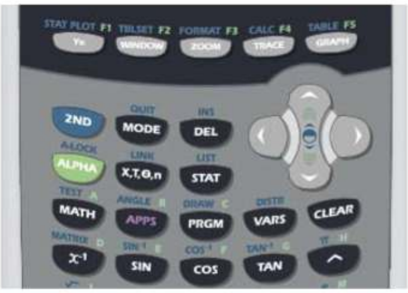

STAT PLOT F1
TBLSET F2
FORMAT F3
CALC F4
TABLE F5
Y=
WINDOW
ZOOM
TRACE
GRAPH
QUIT
INS
2ND
MODE
DEL
A-LOCK
LINK
LIST
ALPHA
X,T,Θ,n
STAT
TEST A
ANGLE B
DRAW C
DISTR
MATH
APPS
PRGM
VARS
CLEAR
MATRIX D
SIN⁻¹ E
COS⁻¹ F
TAN⁻¹ G
π H
x⁻¹
SIN
COS
TAN
^

Figure 3.2.2: Top half of calculator.

It is not uncommon that people share their graphing calculators with friends who make changes to the settings in the calculator. Let’s take a moment to make sure we have some common settings on our calculators.

Note

Above each button are one or more commands located on the calculator’s case. Press the 2ND key to access a command having the same color as the 2ND key. Press the ALPHA key to access a command having the same color as the ALPHA key.

Locate and push the MODE in the first row of Figure 3.2.2. This will open the window shown in Figure 3.2.3. Make sure that the mode settings on your calculator are identical to the ones shown in Figure 3.2.3. If not, use the up-and-down arrow keys to move to the non-matching line item. This should place a blinking cursor over the first item on the line. Press the ENTER button on the lower-right corner of your calculator to make the selection permanent. Once you’ve completed your changes, press 2ND MODE again to quit the MODE menu.

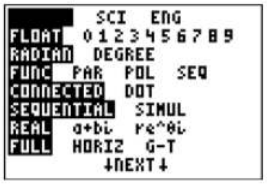

NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi re^θi
FULL HORIZ G-T
↓NEXT↓

Figure $3.2.3$: Settings in the MODE window.

Note

The QUIT command is located above the MODE button on the calculator case and is used to exit the current menu.

Next, note the buttons across the first row of the calculator, located immediately below the view screen.

Y= WINDOW ZOOM TRACE GRAPH

Figure $3.2.2$). Press the 2ND button, then the **Y**= button. This opens the stat plot menu shown in Figure $3.2.4$.

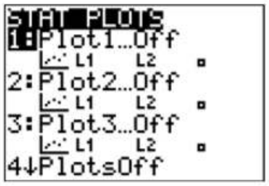

STAT PLOTS
1:Plot1...Off
L1 L2
2:Plot2...Off
L1 L2
3:Plot3...Off
L1 L2
4↓PlotsOff

Figure $3.2.4$: The STAT PLOT menu.

We need all of the stat plots to be “off.” If any of the three stat plots are “on,” select **4:PlotsOff** (press the number 4 on your keyboard), then press the ENTER key on the lower right corner of your calculator. That’s it! Your calculator should now be ready for the upcoming exercises.

Example $3.2.1$

Use the graphing calculator to sketch the graph of $y = x + 1$.

Solution

Recall that we drew the graph of $y = x + 1$ by hand in Example 3.1.1 of Section 3.1 (see Figure 3..1.11). In this example, we will use the graphing calculator to produce the same result.

Press the **Y**= button. The window shown in Figure $3.2.5$ appears. If any equations appear in the **Y**= menu of Figure $3.2.5$, use the up-and-down arrow keys and the CLEAR button (located below the up-and-down and left-and-right arrow keys) to delete them.

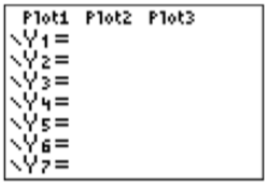

Plot1 Plot2 Plot3
\Y1=
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

Figure $3.2.5$: Press the Y= button to open the Y= menu.

Next, move the cursor to **Y1**=, then enter the equation $y = x + 1$ in **Y1** with the following button keystrokes. The result is shown in Figure $3.2.6$.

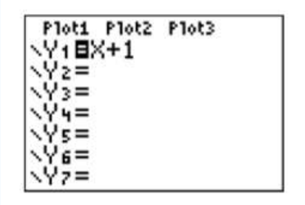

Plot1 Plot2 Plot3
\Y1■X+1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

Figure 3.2.6: Enter $y=x+1$ in **Y1**=.

X,T,θ,n + 1 ENTER

Figure 3.2.7. From the ZOOM menu, select **6:ZStandard**.

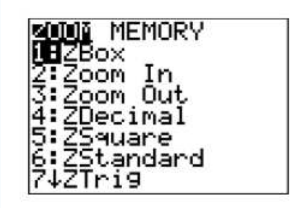

ZOOM MEMORY
1:ZBox
2:Zoom In
3:Zoom Out
4:ZDecimal
5:ZSquare
6:ZStandard
7↓ZTrig

Figure 3.2.7: Select **6:ZStandard** from the **ZOOM** menu.

There are two ways to make this selection:

1. Use the down-arrow key to move downward in the **ZOOM** menu until **6:ZStandard** is highlighted, then press the **ENTER** key.
2. A quicker alternative is to simply press the number 6 on the calculator keyboard to select **6:ZStandard**.

The resulting graph of $y=x+1$ is shown in Figure 3.2.8. Note that the result is identical to the graph of $y=x+1$ drawn by hand in Example 3.1.3 of Section 3.1 (see Figure 3..1.11).

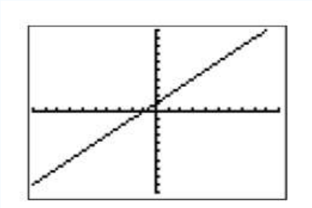
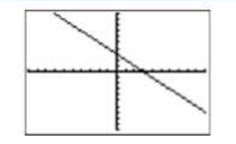


Figure 3.2.8: The graph of $y=x+1$ is a line.

Exercise 3.2.1

Use the graphing calculator to sketch the graph of $y=-x+3$.

Answer



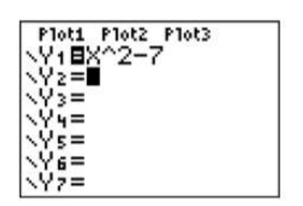
Example 3.2.2

Use the graphing calculator to sketch the graph of $y=x^2-7$.

Solution

Recall that we drew the graph of $y=x^2-7$ by hand in Example 4 of Section 3.1 (see Figure 3.1.22). In this example, we will use the graphing calculator to produce the same result. Press the **Y**= button and enter the equation $y=x^2-7$ into **Y1** (see Figure 3.2.9) with the following keystrokes:

X,T,θ,n **∧** 2 − 7 **ENTER**



```
Plot1 Plot2 Plot3
\Y1=X^2-7
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 3.2.9: Enter $y = x^2 - 7$ in **Y1**.

The caret (∧) symbol (see Figure 3.2.2) is located in the last column of buttons on the calculator, just underneath the CLEAR button, and means "raised to." The caret button is used for entering exponents. For example, x^2 is entered as $\mathrm{X} \wedge 2$, x^3 is entered as $\mathrm{X} \wedge 3$, and so on.

Press the **ZOOM** button, then select **6:ZStandard** from the **ZOOM** menu to produce the graph of $y = x^2 - 7$ shown in Figure 3.2.10. Note that the result in Figure 3.2.10 agrees with the graph of $y = x^2 - 7$ we drew by hand in Example 4 of Section 3.1 (see Figure 3.1.22).

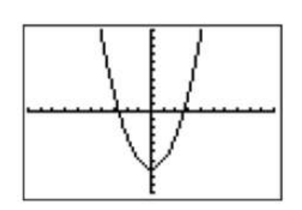
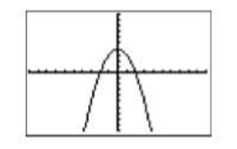


Figure 3.2.10: The graph of $y = x^2 - 7$ is a parabola.

Exercise 3.2.2

Use the graphing calculator to sketch the graph of $y = -x^2 + 4$.

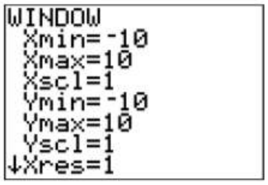
Answer



Reproducing Calculator Results on Homework Paper

In this section we delineate recommendations and requirements when reproducing graphing calculator results on your homework paper.

Consider again the final result of Example 3.2.2 shown in Figure 3.2.10. To determine the scale at each end of each axis, press the WINDOW button on the topmost row of buttons on your calculator. The WINDOW settings for Figure 3.2.10 are shown in Figure 3.2.11. Figure 3.2.12 presents a visual explanation of each of the entries $\mathrm{Xmin}, \mathrm{Xmax}, \mathrm{Ymin}$ and Ymax.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
↓Xres=1
```

Figure 3.2.11: The **WINDOW** parameters.

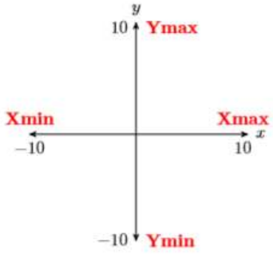


Figure 3.2.12: Xmin, Xmax, Ymin and Ymax contain the scale at the end of each axis.

Xmin and Xmax indicate the scale at the left- and right-hand ends of the x-axis, respectively, while Ymin and Ymax indicate the scale at the bottomand top-ends of the y-axis. Finally, as we shall see shortly, Xscl and Yscl control the spacing between tick marks on the x- and y-axes, respectively.

When reproducing the graph in your calculator viewing window on your homework paper, follow the *Calculator Submission Guidelines*.

Calculator Submission Guidelines

1. All lines (including the horizontal and vertical axes) should be drawn with a ruler. All curves should be drawn freehand.
2. Set up a coordinate system on your homework paper that mimics closely the coordinate system in your calculator's view screen. Label your axes (usually with x and y).
3. Indicate the scale at each end of each axis. Use Xmin, Xmax, Ymin and Ymax in the **WINDOW** menu for this purpose.
4. Copy the graph from your calculator's viewing window onto your coordinate system. Label the graph with its equation.

For example, to report the results of Figure 3.2.10, draw the axes with a ruler, label the horizontal axis with x, the vertical axis with y, then place the values of Xmin, Xmax, Ymin and Ymax at the appropriate end of each axis (see Figure 3.2.13).

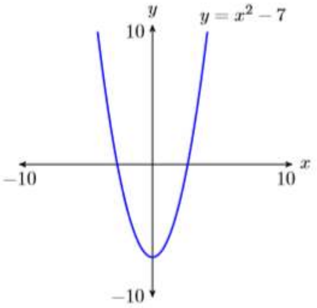


Figure 3.2.13: Reporting the graph of $y = x^2 - 7$ on your homework paper.

Because the graph is a curve, make a freehand copy that closely mimics the graph shown in Figure 3.2.10, then label it with its equation, as shown in Figure 3.2.13.

Adjusting the Viewing Window

Note that Figure 3.2.12 gives us some sense of the meaning of the "standard viewing window" produced by selecting **6:ZStandard** from the ZOOM menu. Each time you select **6:ZStandard** from the ZOOM menu, Xmin is set to -10, Xmax to 10, and Xscl to 1 (distance between the tick marks on the x-axis). Similarly, Ymin is set to -10, Ymax to 10, and Yscl to 1 (distance between the tick marks on the y-axis). You can, however, override these settings as we shall see in the next example.

Example $3.2.3$

Sketch the graphs of the following equations on the same viewing screen.

$$y=\frac{5}{4}x-3 \qquad \text{and} \qquad y=\frac{2}{3}x+4$$

Adjust the viewing window so that the point at which the graphs intersect (where the graphs cross one another) is visible in the viewing window, then use the TRACE button to approximate the coordinates of the point of intersection.

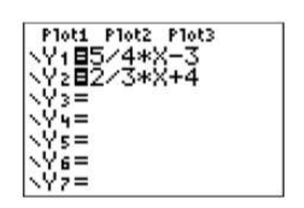
Solution

We must first decide on the proper syntax to use when entering the equations. In the case of the first equation, note that

$$y=\frac{5}{4}x-3$$

is pronounced "y equals five-fourths x minus three," and means "y equals five-fourths times x minus three." Enter this equation into **Y1** in the **Y**= menu as $5/4 * \mathbf{X} - 3$ (see Figure $3.2.14$), using the button keystrokes:

5 ÷ 4 × X, T, θ, n − 3



```
Plot1 Plot2 Plot3
\Y1=5/4*X-3
\Y2=2/3*X+4
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $3.2.14$: Enter equations.

The division and times buttons are located in the rightmost column of the calculuator. Similarly, enter the second equation into **Y2** as $2/3 * X + 4$ using the button keystrokes:

2 ÷ 3 × X, T, θ, n + 4

Figure $3.2.15$.

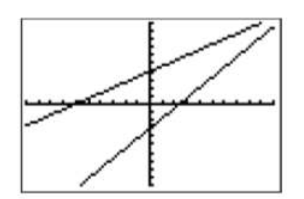
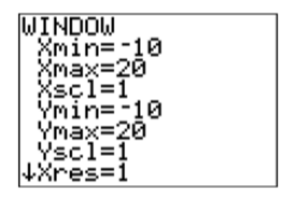


Figure $3.2.15$: The graphs.

When we examine the resulting graphs in Figure $3.2.15$, it appears that their point of intersection will occur off the screen above and to the right of the upper right-corner of the current viewing window. With this thought in mind, let's extend the x-axis to the right by increasing the value of Xmax to 20. Further, let's extend the y-axis upward by increasing the value of Ymin to 20. Press the WINDOW button on the top row of the calculator, then make the adjustments shown in Figure $3.2.16$.



```
WINDOW
 Xmin=-10
 Xmax=20
 Xscl=1
 Ymin=-10
 Ymax=20
 Yscl=1
↓Xres=1
```

Figure $3.2.16$: Adjusting the viewing window.

If we select **6:ZStandard** from the ZOOM menu, our changes to the WINDOW parameters will be discarded and the viewing window will be returned to the "standard viewing window." If we want to keep our changes to the WINDOW parameters, the correct approach at this point is to push the GRAPH button on the top row of the calculator. The resulting graph is shown in Figure 3.2.17. Note that the point of intersection of the two lines is now visible in the viewing window.

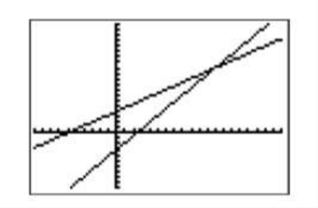
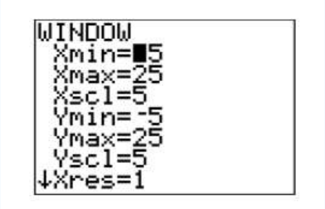


Figure 3.2.17: The point of intersection is visible in the new viewing window.

The image in Figure 3.2.17 is ready for recording onto your homework. However, we think we would have a better picture if we made a couple more changes:

1. It would be nicer if the point of intersection were more centered in the viewing window.
2. There are far too many tick marks.

With these thoughts in mind, make the changes to the WINDOW parameters shown in Figure 3.2.18, then push the GRAPH button to produce the image in Figure 3.2.19.



```
WINDOW
 Xmin=■5
 Xmax=25
 Xscl=5
 Ymin=-5
 Ymax=25
 Yscl=5
↓Xres=1
```

Figure 3.2.18: A final adjustment of the viewing window.

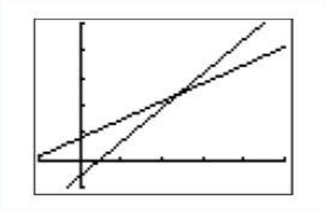
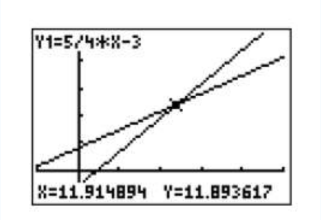


Figure 3.2.19: A centered point of intersection and fewer tick marks.

Finally, push the TRACE button on the top row of buttons, then use the left- and right-arrow keys to move the cursor atop the point of intersection. An approximation of the coordinates of the point of intersection are reported at the bottom of the viewing window (see Figure 3.2.20).



```
Y1=5/4*X-3
X=11.914894  Y=11.893617
```

Figure 3.2.20: Approximate coordinates of the point of intersection reported by the TRACE button.

Note

The TRACE button is only capable of providing a very rough approximation of the point of intersection. In Chapter 4, we’ll introduce the intersection utility on the CALC menu, which will report a much more accurate result.

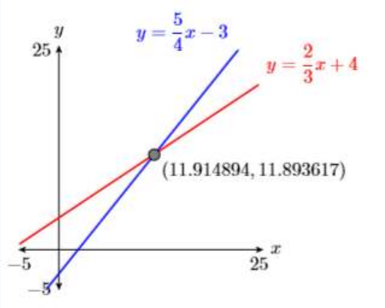


Figure $3.2.21$: Reporting the answer on your homework.

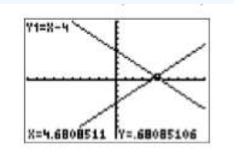
In reporting the answer on our homework paper, note that we followed the *Calculator Submission Guidelines*.

Exercise $3.2.3$

Approximate the point of intersection of the graphs of $y = x - 4$ and $y = 5 - x$ using the TRACE button.

Answer

(4.68,0.68)



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3.3: Rates and Slope

Let’s open this section with an application of the concept of *rate*.

Independent versus dependent

It is traditional to place the independent variable on the horizontal axis and the dependent variable on the vertical axis.

Example $3.3.1$

An object is dropped from rest, then begins to pick up speed at a constant rate of 10 meters per second every second ($10(\mathrm{m/s})/\mathrm{s}$ or $10\mathrm{m/s}^2$). Sketch the graph of the speed of the object versus time.

Solution

In this example, the speed of the object *depends* on the time. This makes the speed the *dependent variable* and time the *independent variable*.

Following this guideline, we place the time on the horizontal axis and the speed on the vertical axis. In Figure $3.3.1$, note that we’ve labeled each axis with the dependent and independent variables (v and t), and we’ve included the units ($\mathrm{m/s}$ and s) in our labels. Next, we need to scale each axis. In determining a scale for each axis, keep two thoughts in mind:

1. Pick a scale that makes it convenient to plot the given data.
2. Pick a scale that allows all of the given data to fit on the graph.

In this example, we want a scale that makes it convenient to show that the speed is increasing at a rate of 10 meters per second ($10\mathrm{m/s}$) every second ($\mathrm{m/s}$). One possible approach is to make each tick mark on the horizontal axis equal to $1\mathrm{s}$ and each tick mark on the vertical axis equal to $10\mathrm{m/s}$.

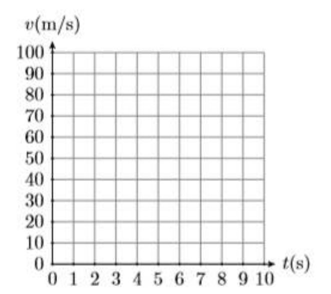


Figure $3.3.1$: Label and scale each axis. Include units with labels.

Next, at time $t = 0\mathrm{s}$, the speed is $v = 0\mathrm{m/s}$. This is the point $(t, v) = (0, 0)$ plotted in Figure $3.3.2$. Secondly, the rate at which the speed is increasing is ($10\mathrm{m/s}$) per second. This means that every time you move 1 second to the right, the speed increases by ($10\mathrm{m/s}$).

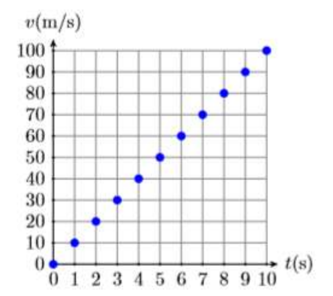


Figure $3.3.2$: Start at $(0, 0)$, then continuously move 1 right and 10 up.

In Figure $3.3.2$, start at $(0, 0)$, then move $1\mathrm{s}$ to the right and ($10\mathrm{m/s}$) up. This places you at the point $(1, 10)$, which says that after 1 second, the speed of the particle is ($10\mathrm{m/s}$) . Continue in this manner, continuously moving $1\mathrm{s}$ to the right and ($10\mathrm{m/s}$) upward. This produces the sequence of points shown in Figure $3.3.2$. Note that this constant rate of $10(\mathrm{m/s})/\mathrm{s}$ forces the graph of the speed versus time to be a line, as depicted in Figure $3.3.3$.

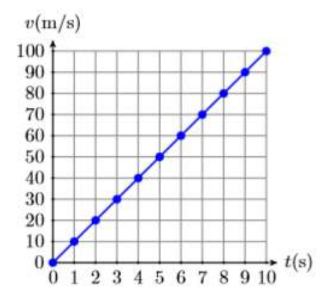
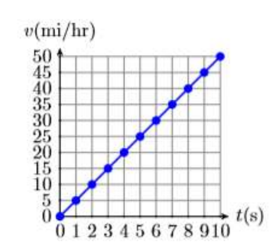


Figure 3.3.3: The constant rate forces the graph to be a line.

Exercise 3.3.1

Starting from rest, an automobile picks up speed at a constant rate of 5 miles per hour every second (5\((\mathrm{mi} / \mathrm{hr}) / \mathrm{s}\)). Sketch the graph of the speed of the object versus time.

Answer



Measuring the Change in a Variable

To calculate the change in some quantity, we take a difference. For example, suppose that the temperature in the morning is $40^{\circ}\mathrm{F}$, then in the afternoon the temperature measures $60^{\circ}\mathrm{F}$ (F stands for Fahrenheit temperature). Then the change in temperature is found by taking a difference.

$$\begin{aligned} \text{Change in temperature } &= \text{ Afternoon temperature - Morning temperature} \\ &= 60^{\circ}\mathrm{F} - 40^{\circ}\mathrm{F} \\ &= 20^{\circ}\mathrm{F} \end{aligned}$$

Therefore, there was a twenty degree increase in temperature from morning to afternoon.

Now, suppose that the evening temperature measures $50^{\circ}\mathrm{F}$. To calculate the change in temperature from the afternoon to the evening, we again subtract.

$$\begin{aligned} \text{Change in temperature } &= \text{ Evening temperature - Afternoon temperature} \\ &= 50^{\circ}\mathrm{F} - 60^{\circ}\mathrm{F} \\ &= -10^{\circ}\mathrm{F} \end{aligned}$$

There was a ten degree decrease in temperature from afternoon to evening.

Calculating the Change in a Quantity

To calculate the change in a quantity, subtract the earlier measurement from the later measurement.

Let T represent the temperature. Mathematicians like to use the symbolism ΔT to represent the change in temperature. For the change in temperature from morning to afternoon, we would write $\Delta T = 20^{\circ}\mathrm{F}$. For the afternoon to evening change, we would write $\Delta T = -10^{\circ}\mathrm{F}$.

Mathematicians and scientists make frequent use of the Greek alphabet, the first few letters of which are:

$$\begin{array}{ll} \alpha, \beta, \gamma, \delta, \ldots & \text{(Greek alphabet, lower case)} \\ A, B, \Gamma, \Delta, \ldots & \text{(Greek alphabet, upper case)} \\ a, b, c, d, \ldots & \text{(English alphabet)} \end{array}$$

Thus, the Greek letter ΔT, the upper case form of δ, correlates with the letter ‘d’ in the English alphabet. Why did mathematicians make this choice of letter to represent the change in a quantity? Because to find the change in a quantity, we take a difference, and the word “difference” starts with the letter ‘d.’ Thus, ΔT is also pronounced “the difference in T.”

Important Pronunciations

Two ways to pronounce the symbolism ΔT.

1. ΔT is pronounced “the change in T.”
2. ΔT is also pronounced “the difference in T.”

Slope as Rate

Here is the definition of the slope of a line.

Slope

The slope of a line is the rate at which the dependent variable is changing with respect to the independent variable. For example, if the dependent variable is y and the independent variable is x, then the slope of the line is:

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

Example 3.3.2

In Example 3.3.1, an object released from rest saw that its speed increased at a constant rate of 10 meters per second per second ($10(\mathrm{m/s})/\mathrm{s}$ or $10\mathrm{m/s}^2$). This constant rate forced the graph of the speed versus time to be a line, shown in Figure 3.3.3. Calculate the slope of this line.

Solution

Start by selecting two points $P(2,20)$ and $Q(8,80)$ on the line, as shown in Figure 3.3.4. To find the slope of this line, the definition requires that we find the rate at which the dependent variable v changes with respect to the independent variable t. That is, the slope is the change in v divided by the change in t. In symbols:

$$\text{Slope} = \frac{\Delta v}{\Delta t}$$

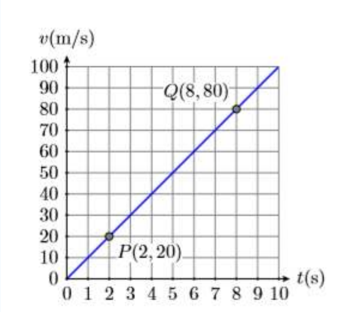


Figure 3.3.4: Pick two points to compute the slope.

Now, as we move from point $P(2,20)$ to point $Q(8,80)$, the speed changes from $20\mathrm{m/s}$ to $80\mathrm{m/s}$. Thus, the change in the speed is:

$$\begin{aligned} \Delta v &= 80\mathrm{m/s} - 20\mathrm{m/s} \\ &= 60\mathrm{m/s} \end{aligned}$$

Similarly, as we move from point P(2,20) to point Q(8,80), the time changes from 2 seconds to 8 seconds. Thus, the change in time is:

$$\begin{aligned}\Delta t &= 8\text{s} - 2\text{s}\\ &= 6\text{s}\end{aligned}$$

Now that we have both the change in the dependent and independent variables, we can calculate the slope.

$$\begin{aligned}\text{Slope } &= \frac{\Delta v}{\Delta t}\\ &= \frac{60\text{m/s}}{6\text{s}}\\ &= 10\frac{\text{m/s}}{\text{s}}\end{aligned}$$

Therefore, the slope of the line is 10 meters per second per second ($10(\text{m/s})/\text{s}$ or 10m/s^2).

The slope of a line does not depend upon the points you select. Let's try the slope calculation again, using two different points and a more compact presentation of the required calculations. Pick points $P(3,30)$ and $Q(7,70)$ as shown in Figure $3.3.5$. Using these two new points, the slope is the rate at which the dependent variable v changes with respect to the independent variable t.

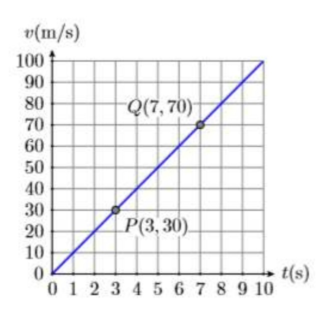


Figure $3.3.5$: The slope does not depend on the points we select on the line.

$$\begin{aligned}\text{Slope } &= \frac{\Delta v}{\Delta t}\\ &= \frac{70\text{m/s} - 30\text{m/s}}{7\text{s} - 3\text{s}}\\ &= \frac{40\text{m/s}}{4\text{s}}\\ &= 10\frac{\text{m/s}}{\text{s}}\end{aligned}$$

Again, the slope of the line is $10(\text{m/s})/\text{s}$.

Exercise $3.3.2$

Starting from rest, an automobile picks up speed at a constant rate of 5 miles per hour every second (5(mi/hr)/s). The constant rate forces the graph of the speed of the object versus time to be a line. Calculate the slope of this line.

Answer

$5(\text{mi/hr})/\text{s}$

Example \(\PageIndex{2}\) points out the following fact.

Slope is independent of the selected points

It does not matter which two points you pick on the line to calculate its slope.

The next example demonstrates that the slope is also independent of the order of subtraction.

Example 3.3.3

Compute the slope of the line passing through the points $P(-1,-2)$ and $Q(3,3)$.

Solution

First, sketch the line passing through the points P(−1,−2) and Q(3,3) (see Figure \(\PageIndex{6}\)).

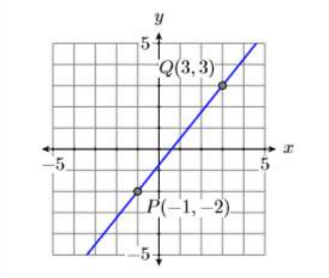


Figure 3.3.6: Computing the slope of the line passing through the points $P(-1,-2)$ and $Q(3,3)$.

To calculate the slope of the line through the points $P(-1,-2)$ and $Q(3,3)$, we must calculate the change in both the independent and dependent variables. We'll do this in two different ways.

Warning!

If you are not consistent in the direction you subtract, you will not get the correct answer for the slope.

For example:

$$\frac{3-(-2)}{-1-3}=-\frac{5}{4}$$

In this case, we subtracted the y-coordinate of point $P(-1,-2)$ from the y-coordinate of point $Q(3,3)$, but then we changed horses in midstream, subtracting the x-coordinate of point $Q(3,3)$ from the x-coordinate of point $P(-1,-2)$. Note that we get the negative of the correct answer.

Method 1

Subtract the coordinates of point $P(-1,-2)$ from the coordinates of point $Q(3,3)$.

$$\begin{aligned}\text{Slope } &= \frac{\Delta y}{\Delta x}\\ &= \frac{3-(-2)}{3-(-1)}\\ &= \frac{5}{4}\end{aligned}$$

Method 2

Subtract the coordinates of point $Q(3,3)$ from the coordinates of point $P(-1,-2)$.

$$\begin{aligned}\text{Slope } &= \frac{\Delta y}{\Delta x}\\ &= \frac{-2-3}{-1-3}\\ &= \frac{-5}{-4}\\ &= \frac{5}{4}\end{aligned}$$

Note that regardless of the direction of subtraction, the slope is $5/4$.

Exercise 3.3.3

Compute the slope of the line passing through the points P(−3,1) and Q(2,4).

Answer

$3/5$

Example 3.3.3 demonstrates the following fact.

The direction of subtraction does not matter

When calculating the slope of a line through two points P and Q, it does not matter which way you subtract, provided you remain consistent in your choice of direction.

The Steepness of a Line

We need to examine whether our definition of slope matches certain expectations.

Slope and steepness of a line

The slope of a line is a number that tells us how steep the line is.

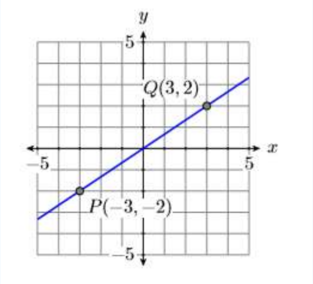
If slope is a number that measures the steepness of a line, then one would expect that a steeper line would have a larger slope.

Example 3.3.4

Graph two lines, the first passing through the points $P(-3,-2)$ and $Q(3,2)$ and the second through the points $R(-1,-3)$ and $S(1,3)$. Calculate the slope of each line and compare the results.

Solution

The graphs of the two lines through the given points are shown, the first in Figure 3.3.7 and the second in Figure 3.3.8. Note that the line in Figure 3.3.7 is less steep than the line in Figure 3.3.8.



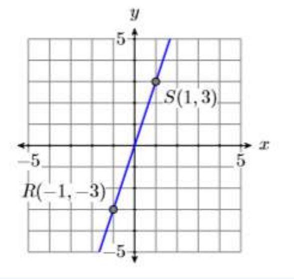


Figure 3.3.7: This line is less steep than the line on the right. **Figure** 3.3.8**:** This line is steeper than the line on the left.

Remember, the slope of the line is the rate at which the dependent variable is changing with respect to the independent variable. In both Figure 3.3.7 and Figure 3.3.8, the dependent variable is y and the independent variable is x.

Subtract the coordinates of point $P(-3,-2)$ from the coordinates of point $Q(3,2)$.

$$\text{Slope of first line} = \Delta y$$

cc by nc nd

$$\begin{aligned}\text{Slope of first line} &= \frac{}{\Delta x}\\ &= \frac{2-(-2)}{3-(-3)}\\ &= \frac{4}{6}\\ &= \frac{2}{3}\end{aligned}$$

Subtract the coordinates of the point $R(-1,-3)$ from the point $S(1,3)$.

$$\begin{aligned}\text{Slope of second line} &= \frac{\Delta y}{\Delta x}\\ &= \frac{3-(-3)}{1-(-1)}\\ &= \frac{6}{2}\\ &= 3\end{aligned}$$

Note that both lines go uphill and both have positive slopes. Also, note that the slope of the second line is greater than the slope of the first line. This is consistent with the fact that the second line is steeper than the first.

Exercise 3.3.4

Compute the slope of the line passing through the points $P(-2,-3)$ and $Q(2,5)$. Then compute the slope of the line passing through the points $R(-2,-1)$ and $S(5,3)$, and compare the two slopes. Which line is steeper?

Answer

The first line has slope 2, and the second line has slope $4/7$. The first line is steeper.

In Example 3.3.4, both lines slanted uphill and both had positive slopes, the steeper of the two lines having the larger slope. Let's now look at two lines that slant downhill.

Example 3.3.5

Graph two lines, the first passing through the points $P(-3,1)$ and $Q(3,-1)$ and the second through the points $R(-2,4)$ and $S(2,-4)$. Calculate the slope of each line and compare the results.

Solution

The graphs of the two lines through the given points are shown, the first in Figure 3.3.9 and the second in Figure 3.3.10. Note that the line in Figure 3.3.9 goes downhill less quickly than the line in Figure 3.3.10. Remember, the slope of the line is the rate at which the dependent variable is changing with respect to the independent variable. In both Figure 3.3.9 and Figure 3.3.10, the dependent variable is y and the independent variable is x.

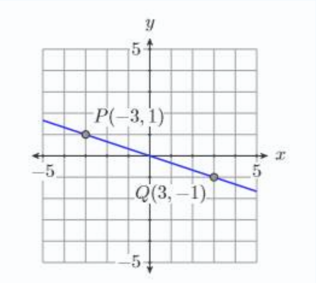


Figure 3.3.9: This line goes downhill more slowly than the line on the right.

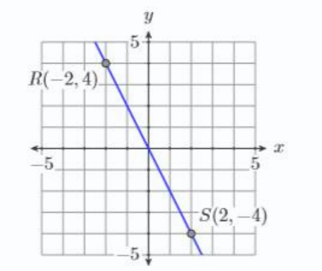


Figure 3.3.10: This line goes downhill more quickly than the line on the left.

Subtract the coordinates of point $P(-3,1)$ from the coordinates of point $Q(3,-1)$.

$$\begin{aligned}\text{Slope of first line} &= \frac{\Delta y}{\Delta x}\\ &= \frac{-1-1}{3-(-3)}\\ &= \frac{-2}{6}\\ &= -\frac{1}{3}\end{aligned}$$

Subtract the coordinates of point $R(-2,4)$ from the coordinates of point $S(2,-4)$.

$$\begin{aligned}\text{Slope of second line} &= \frac{\Delta y}{\Delta x}\\ &= \frac{-4-4}{2-(-2)}\\ &= \frac{-8}{4}\\ &= -2\end{aligned}$$

Note that both lines go downhill and both have negative slopes. Also, note that the magnitude (absolute value) of the slope of the second line is greater than the magnitude of the slope of the first line. This is consistent with the fact that the second line moves downhill more quickly than the first.

Exercise 3.3.5

Compute the slope of the line passing through the points $P(-3,3)$ and $Q(3,-5)$. Then compute the slope of the line passing through the points $R(-4,1)$ and $S(4,-3)$, and compare the two slopes. Which line is steeper?

Answer

The first line has slope $-4/3$, and the second line has slope $-1/2$. The first line is steeper.

What about the slopes of vertical and horizontal lines?

Example 3.3.6

Calculate the slopes of the vertical and horizontal lines passing through the point $(2,3)$.

Solution

First draw a sketch of the vertical and horizontal lines passing through the point (2,3). Next, select a second point on each line as shown in Figures 3.3.11 and 3.3.12.

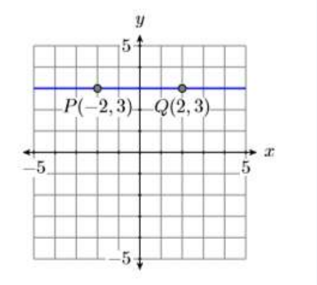


Figure 3.3.11: A horizontal line through $(2, 3)$.

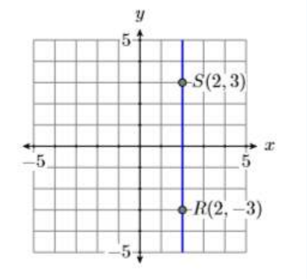


Figure 3.3.12: A vertical line through $(2, 3)$.

The slopes of the horizontal and vertical lines are calculated as follows.

Subtract the coordinates of the point $P(-2, 3)$ from the coordinates of the point $Q(2, 3)$.

$$\begin{aligned}\text{Slope of horizontal line } &= \frac{\Delta y}{\Delta x} \\ &= \frac{3-3}{2-(-2)} \\ &= \frac{0}{4} \\ &= 0\end{aligned}$$

Thus, the slope of the horizontal line is zero, which makes sense because a horizontal line neither goes uphill nor downhill.

Subtract the coordinates of the point $(2, -3)$ from the coordinates of the point $S(2, 3)$.

$$\begin{aligned}\text{Slope of vertical line } &= \frac{\Delta y}{\Delta x} \\ &= \frac{3-(-3)}{2-2} \\ &= \frac{6}{0} \\ &= \text{ undefined}\end{aligned}$$

Division by zero is undefined. Hence, the slope of a vertical line is undefined. Again, this makes sense because as uphill lines get steeper and steeper, their slopes increase without bound.

Exercise 3.3.6

Calculate the slopes of the vertical and horizontal lines passing through the point $(-4, 1)$.

Answer

The slope of the vertical line is undefined. The slope of the second line is 0.

The Geometry of the Slope of a Line

We begin our geometrical discussion of the slope of a line with an example, calculating the slope of a line passing through the points $P(2, 3)$ and $Q(8, 8)$. Before we begin we'll first calculate the change in y and the change in x by subtracting the coordinates of point $P(2,3)$ from the coordinates of point $Q(8,8)$.

$$\begin{aligned}\text{Slope} &= \frac{\Delta y}{\frac{\Delta x}{\Delta x}} \\ &= \frac{8-3}{8-2} \\ &= \frac{5}{6}\end{aligned}$$

Thus, the slope of the line through the points $P(2,3)$ and $Q(8,8)$ is $5/6$.

To use a geometric approach to finding the slope of the line, first draw the line through the points $P(2,3)$ and $Q(8,8)$ (see Figure $3.3.13$). Next, draw a right triangle with sides parallel to the horizontal and vertical axes, using the points $P(2,3)$ and $Q(8,8)$ as vertices. As you move from point P to point R in Figure $3.3.13$, note that the change in x is $\Delta x = 6$ (count the tick marks[1]).

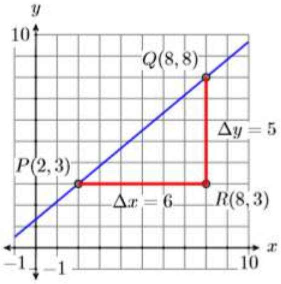


Figure $3.3.13$: Determining the slope of the line from the graph.

As you then move from point R to point Q, the change in y is $\Delta y = 5$ (count the tick marks). Thus, the slope is $\Delta y/\Delta x = 5/6$, precisely what we got in the previous computation.

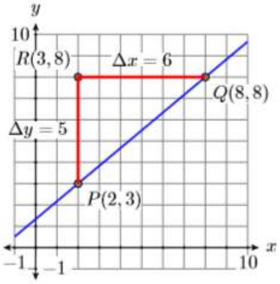


Figure $3.3.14$: Determining the slope of the line from the graph.

For contrast, in Figure $3.3.14$, we started at the point $P(2,3)$, then moved upward 5 units and right 6 units. However, the change in y is still $\Delta y = 5$ and the change in x is still $\Delta x = 6$ as we move from point $P(2,3)$ to point $Q(8,8)$. Hence, the slope is still $\Delta y/\Delta x = 5/6$.

Rise over run

In Figure $3.3.14$, we start at the point $P(2,3)$, then “rise” 5 units, then “run” 6 units to the right. For this reason, some like to think of the slope as “rise over run.”

Consider a second example shown in Figure $3.3.15$. Note that the line slants downhill, so we expect the slope to be a negative number.

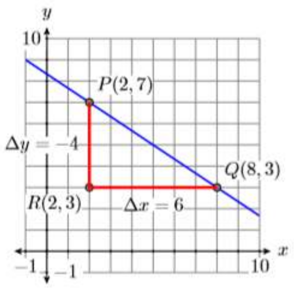


Figure 3.3.15: Determining the slope of the line from the graph.

In Figure 3.3.15, we've drawn a right triangle with sides parallel to the horizontal and vertical axes, using the points $P(2, 7)$ and $Q(8, 3)$ as vertices. As you move from point P to point R in Figure 3.3.15, the change in y is $\Delta y = -4$ (count the tick marks and note that your values of y are decreasing as you move from P to R). As you move from point R to point Q, the change is x is $\Delta x = 6$ (count the tick marks and note that your values of x are increasing as you move from R to Q). In this case, the "rise" is negative, while the "run" is positive.

Thus, the slope is $\Delta y/\Delta x = -4/6$, or $-2/3$. Note that the slope is negative, as anticipated.

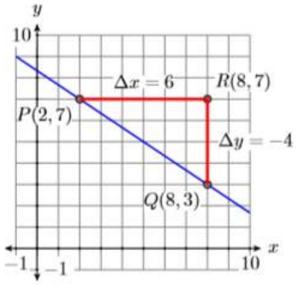


Figure 3.3.16: Determining the slope of the line from the graph.

In Figure 3.3.16, we've drawn our triangle on the opposite side of the line. In this case, as you move from point P to point R in Figure 3.3.16, the change in x is $\Delta x = 6$ (count the tick marks and note that your values of x are increasing as you move from P to R). As you move from point R to point Q, the change is y is $\Delta y = -4$ (count the tick marks and note that your values of y are decreasing as you move from R to Q). Thus, the slope is still $\Delta y/\Delta x = -4/6$, or $-2/3$.

We can verify our geometrical calculations of the slope by subtracting the coordinates of the point $P(2, 7)$ from the point $Q(8, 3)$.

$$\begin{aligned} \text{Slope} &= \frac{\Delta y}{\Delta x} \\ &= \frac{3-7}{8-2} \\ &= \frac{-4}{6} \\ &= -\frac{2}{3} \end{aligned}$$

This agrees with the calculations made in Figures 3.3.15 and 3.3.16.

Let's look at a final example.

Example 3.3.7

Sketch the line passing through the point $(-2, 3)$ with slope $-2/3$.

Solution

The slope is $-2/3$, so the line must go downhill. In Figure 3.3.17, we start at the point $P(-2, 3)$, move right 3 units to the point $R(1, 3)$, then move down 2 units to the point $Q(1, 1)$. Draw the line through the points P and Q and you are done.

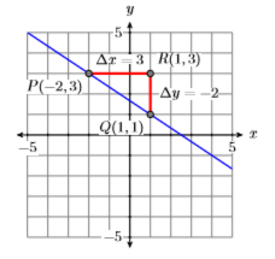


Figure 3.3.17: Start at $P(-2, 3)$, then move right 3 and down 2. The resulting line has slope $-2/3$.

In Figure $3.3.18$, we take a different approach that results in the same line. Start at the point $P(-2, 3)$, move downward 4 units to the point $R(-2, -1)$, then right 6 units to the point $Q(4, -1)$. Draw a line through the points P and Q and you are done.

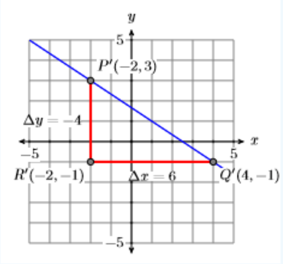


Figure 3.3.18: Starting at $P(-2, 3)$ and moving down 4 and right 6 also yields a slope of $-2/3$.

The triangle PQR in Figure $3.3.17$ is similar to the triangle PQR in Figure $3.3.18$, so their sides are proportional. Consequently, the slope of the line through points $P(-2, 3)$ and $Q(4, -1)$,

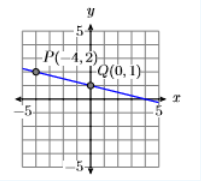
$$\begin{aligned}\text{Slope } &= \frac{\Delta y}{\Delta x}\\ &= \frac{-4}{6}\\ &= -\frac{2}{3}\end{aligned}$$

reduces to the slope of the line through the points P and Q in Figure $3.3.17$.

Exercise $3.3.7$

Sketch the line passing through the point $(-4, 2)$ with slope $-1/4$.

Answer



A summary of facts about the slope of a line

We present a summary of facts learned in this section.

1. The slope of a line is the rate at which the dependent variable is changing with respect to the independent variable. If y is the dependent variable and x is the independent variable, then the slope is

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

where Δy is the change in y (difference in y) and Δx is the change in x (difference in x).

2. If a line has positive slope, then the line slants uphill as you “sweep your eyes from left to right.” If two lines have positive slope, then the line with the larger slope rises more quickly.
3. If a line has negative slope, then the line slants downhill as you “sweep your eyes from left to right.” If two lines have negative slope, then the line having the slope with the larger magnitude falls more quickly.
4. Horizontal lines have slope zero.
5. Vertical lines have undefined slope.

References

[1] When counting tick marks, make sure you know the amount each tick mark represents. For example, if each tick mark represents two units, and you count six tick marks when evaluating the change in x, then $\Delta x = 12$.

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3.4: Slope-Intercept Form of a Line

We start with the definition of the y-intercept of a line.

The y-intercept

The point $(0, b)$ where the graph of a line crosses the y-axis is called the y-intercept of the line.

We will now generate the slope-intercept formula for a line having y-intercept $(0, b)$ and $\text{Slope} = m$ (see Figure $3.4.1$). Let (x, y) be an arbitrary point on the line.

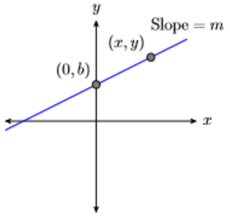


Figure $3.4.1$: Line with y-intercept at $(0, b)$ and $\text{Slope} = m$.

Start with the fact that the slope of the line is the rate at which the dependent variable is changing with respect to the independent variable.

$$\text{Slope} = \frac{\Delta y}{\Delta x} \qquad \text{Slope formula.}$$

Substitute m for the slope. To determine both the change in y and the change in x, subtract the coordinates of the point $(0, b)$ from the point (x, y).

$$\begin{aligned} m &= \frac{y-b}{x-0} && \text{Substitute } m \text{ for the Slope. } \Delta y = y - b \quad \text{and } \Delta x = x - 0. \\ m &= \frac{y-b}{x} && \text{Simplify.} \end{aligned}$$

Clear fractions from the equation by multiplying both sides by the common denominator.

$$\begin{aligned} mx &= \left[\frac{y-b}{x}\right] x && \text{Multiply both sides by } x \\ mx &= y - b && \text{Cancel.} \end{aligned}$$

Solve for y.

$$\begin{aligned} mx + b &= y - b + b && \text{Add } b \text{ to both sides.} \\ mx + b &= y && \text{Simplify.} \end{aligned}$$

Thus, the equation of the line is $y = mx + b$.

The Slope-Intercept form of a line

The equation of the line having y intercept $(0, b)$ and slope m is:

$$y = mx + b$$

Because this form of a line depends on knowing the slope m and the intercept $(0, b)$, this form is called the *slope-intercept* form of a line.

Example $3.4.1$

Sketch the graph of the line having equation $y = \frac{3}{5}x + 1$.

Solution

Compare the equation $y=\frac{3}{5}x+1$ with the slope-intercept form $y=mx+b$, and note that $m=3/5$ and $b=1$. This means that the slope is $3/5$ and the y-intercept is $(0,1)$. Start by plotting the y-intercept $(0,1)$, then move uward 3 units and right 5 units, arriving at the point $(5,4)$. Draw the line through the points $(0,1)$ and $(5,4)$, then label it with its equation $y=\frac{3}{5}x+1$.

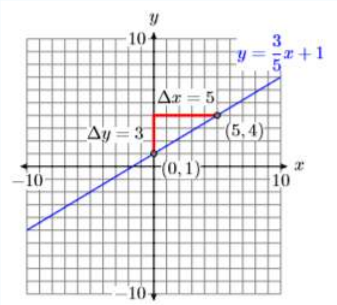


Figure $3.4.2$: Hand-drawn graph of $y=\frac{3}{5}x+1$.

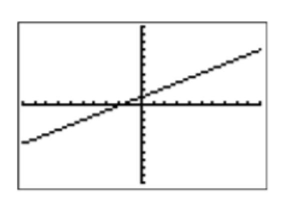


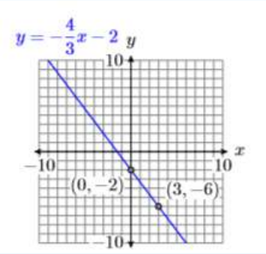
Figure $3.4.3$: Select **5:ZSquare** from the ZOOM menu to draw the graph of $y=\frac{3}{5}x+1$.

When we compare the calculator image in Figure $3.4.3$ with the hand-drawn graph in Figure $3.4.2$, we get a better match.

Exercise $3.4.1$

Sketch the graph of the line having equation $y=-\frac{4}{3}x-2$.

Answer



Example $3.4.2$

Sketch the line with y-intercept $(0,2)$ and slope $-7/3$. Label the line with the slope-intercept form of its equation.

Solution

Plot the y-intercept $(0,2)$. Now use the slope $-7/3$. Start at $(0,2)$, then move down seven units, followed by a three unit move to the right to the point $(3,-5)$. Draw the line through the points $(0,2)$ and $(3,-5)$. (See Figure $3.4.4$).

Next, the y-intercept is $(0,2)$, so $b=2$. Further, the slope is $-7/3$, so $m=-7/3$. Substitute these numbers into the slope-intercept form of the line.

$$\begin{aligned} y&=mx+b && \text{Slope-intercept form.}\\ y&=-\frac{7}{3}x+2 && \text{Substitute: } -7/3 \text{ for } m, 2 \text{ for } b. \end{aligned}$$

Therefore, the slope-intercept form of the line is $y=-\frac{7}{3}x+2$. Label the line with this equation.

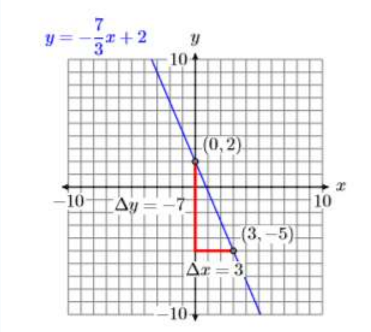


Figure 3.4.4: Hand-drawn graph of $y = -\frac{7}{3}x + 2$

Check: To graph $y = -\frac{7}{3}x + 2$, enter $-7/3 * X + 2$ in Y1 in the **Y=** menu. Select **6:ZStandard** from the ZOOM menu, followed by **5:ZSquare** from the ZOOM menu to produce the graph shown in Figure 3.4.6.

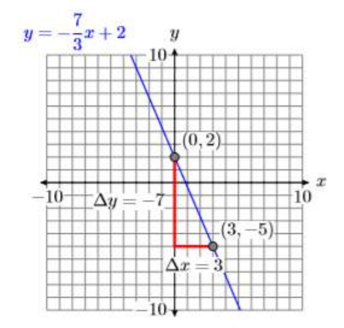


Figure 3.4.5: Hand-drawn graph of $y = -\frac{7}{3}x + 2$

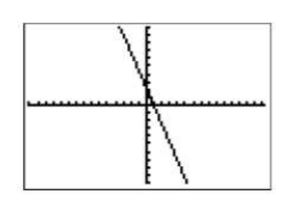


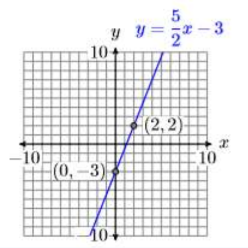
Figure 3.4.6: Select **6:ZStandard** from the ZOOM menu, followed by **5:ZSquare** from the ZOOM menu to produce the graph of the equation $y = -\frac{7}{3}x + 2$

This provides a good match of the hand-drawn graph in Figure 3.4.5 and our graphing calculator result in Figure 3.4.6.

Exercise 3.4.2

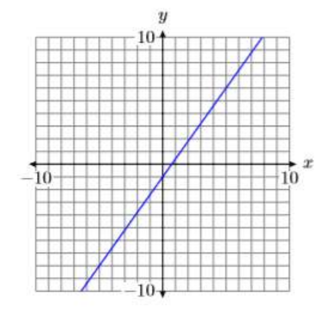
Sketch the line with y-intercept $(0, -3)$ and slope $5/2$. Label the line with the slope-intercept form of its equation.

Answer



Example 3.4.3

Use the graph of the line in the following figure to find the equation of the line.



Solution

Note that the y-intercept of the line is $(0,-1)$ (See Figure $3.4.7$). Next, we try to locate a point on the line that passes directly through a lattice point, a point where a vertical and horizontal grid line intersect. In Figure $3.4.7$, we chose the point $(5,6)$. Now, starting at the y-intercept $(0,1)$, we move up 7 units, then to the right 5 units. Hence, the slope is $m=\Delta y/\Delta x$, or $m=7/5$.

Note

We could also subtract the coordinates of point $(0,-1)$ from the coordinates of point $(5,6)$ to determine the slope.

$$\frac{6-(-1)}{5-0}=\frac{7}{5}$$

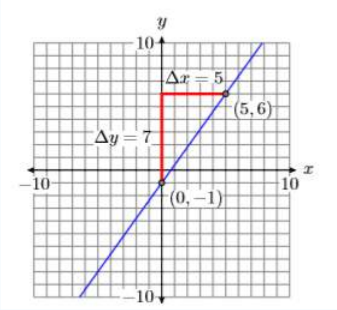


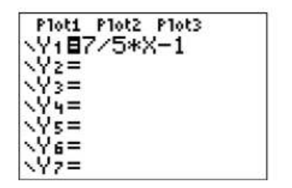
Figure $3.4.7$: The line has y-intercept $(0,-1)$ and slope $7/5$.

Next, state the slope-intercept form, the substitute $7/5$ for m and -1 for b.

$$\begin{aligned} y &= mx+b && \text{Slope-intercept form.} \\ y &= \frac{7}{5}x+(-1) && \text{Substitute: } 7/5 \text{ for } m, -1 \text{ for } b \end{aligned}$$

Thus, the equation of the line is $y=\frac{7}{5}x-1$.

Check: This is an excellent situation to run a check on your graphing calculator.



```
Plot1 Plot2 Plot3
\Y1■7/5*X-1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $3.4.8$: Enter $y=\frac{7}{5}x-1$ in the **Y**= menu.

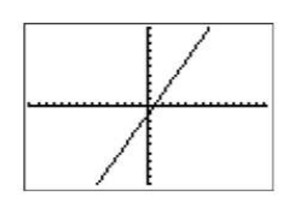
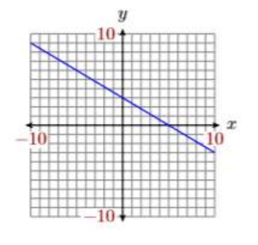


Figure 3.4.9: Select **6:ZStandard** followed by **5:ZSquare** (both from the ZOOM menu) to produce this graph.

When we compare the result in Figure 3.4.9 with the original hand-drawn graph (see Figure 3.4.7), we're confident we have a good match.

Exercise 3.4.3

Use the graph of the line in the figure below to find the equation of the line.



Answer

$$y=-\frac{3}{5}x+3$$

Applications

Let's look at a linear application.

Example 3.4.4

Jason spots his brother Tim talking with friends at the library, located 300 feet away. He begins walking towards his brother at a constant rate of 2 feet per second (2 ft/s).

a. Express the distance d between Jason and his brother Tim in terms of the time t.
b. At what time after Jason begins walking towards Tim are the brothers 200 feet apart?

Solution

Because the distance between Jason and his brother is decreasing at a constant rate, the graph of the distance versus time is a line. Let's begin by making a rough sketch of the line. In Figure 3.4.10, note that we've labeled what are normally the x- and y-axes with the time t and distance d, and we've included the units with our labels.

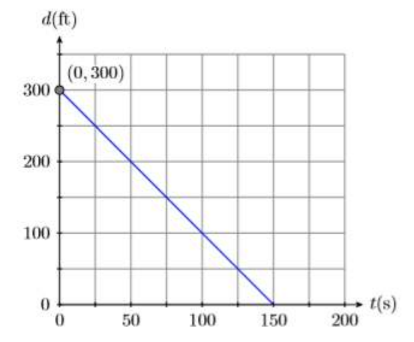


Figure 3.4.10: A plot of the distance d separating the brothers versus time t.

Let $t = 0$ seconds be the time that Jason begins walking towards his brother Tim. At time $t = 0$, the initial distance between the brothers is 300 feet. This puts the d-intercept (normally the y-intercept) at the point $(0, 300)$ (see Figure $3.4.10$).

Because Jason is walking toward his brother, the distance between the brothers decreases at a constant rate of 2 feet per second. This means the line must slant downhill, making the slope negative, so $m = -2$ ft/s. We can construct an accurate plot of distance versus time by starting at the point $(0, 300)$, then descending $\Delta d = -300$, then moving to the right $\Delta t = 150$. This makes the slope $\Delta d/\Delta t = -300/150 = -2$ (See Figure $3.4.10$). Note that the slope is the rate at which the distance d between the brothers is changing with respect to time t.

Finally, the equation of the line is $y = mx + b$, where m is the slope of the line and b is the y-coordinate (in this case the d-coordinate) of the point where the graph crosses the vertical axis. Thus, substitute -2 for m, and 300 for b in the slope-intercept form of the line.

$$\begin{aligned} y &= mx + b && \color{red}{\text{Slope-intercept form.}} \\ y &= -2x + 300 && \color{red}{\text{Substitute: } -2 \text{ for } m, 300 \text{ for } b} \end{aligned}$$

One problem remains. The equation $y = -2x + 300$ gives us y in terms of x.

a. The question required that we express the distance d in terms of the time t. So, to finish the solution, replace y with d and x with t (check the axes labels in Figure $3.4.10$) to obtain a solution:

$$d = -2t + 300$$

b. Now that our equation expresses the distance between the brothers in terms of time, let's answer part (b), "At what time after Jason begins walking towards Tim are the brothers 200 feet apart?" To find this time, substitute 200 for d in the equation $d = -2t + 300$, then solve for t.

$$\begin{aligned} d &= -2t + 300 && \color{red}{\text{Distance equation}} \\ 200 &= -2t + 300 && \color{red}{\text{Substitute 200 for } d} \end{aligned}$$

Solve this last equation for the time t.

$$\begin{aligned} 200 - 300 &= -2t + 300 - 300 && \color{red}{\text{Subtract 300 from both sides.}} \\ -100 &= -2t && \color{red}{\text{Simplify both sides.}} \\ \frac{-100}{-2} &= \frac{-2t}{-2} && \color{red}{\text{Divide both sides by } -2} \\ 50 &= t && \color{red}{\text{Simplify both sides.}} \end{aligned}$$

Thus, it takes Jason 50 seconds to close the distance between the brothers to 200 feet.

Exercise $3.4.4$

A swimmer is 50 feet from the beach, and then begins swimming away from the beach at a constant rate of 1.5 feet per second (1.5 ft/s). Express the distance d between the swimmer and the beach in terms of the time t.

Answer

$d = 1.5t + 50$

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3.5: Point-Slope Form of a Line

In the previous section we learned that if we are provided with the slope of a line and its y-intercept, then the equation of the line is $y = mx + b$, where m is the slope of the line and b is the y-coordinate of the line's y-intercept.

However, suppose that the y-intercept is unknown? Instead, suppose that we are given a point (x_0, y_0) on the line and we're also told that the slope of the line is m (see Figure $3.5.1$).

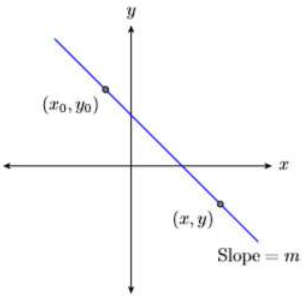


Figure $3.5.1$: A line through the point (x_0, y_0) with slope m.

Let (x, y) be an arbitrary point on the line, then use the points (x_0, y_0) and (x, y) to calculate the slope of the line.

$$\begin{aligned} \text{Slope } &= \frac{\Delta y}{\Delta x} \quad \text{The slope formula.} \\ m &= \frac{y - y_0}{x - x_0} \quad \text{Substitute } m \text{ for the slope. Use } (x, y) \text{ and } (x_0, y_0) \text{ to calculate the} \\ & \text{difference in } y \text{ and the difference in } x \end{aligned}$$

Clear the fractions from the equation by multiplying both sides by the common denominator.

$$\begin{aligned} m(x - x_0) &= \left[\frac{y - y_0}{x - x_0}\right] x - x_0 \quad \text{Multiply both sides by } x - x_0 \\ m(x - x_0) &= y - y_0 \quad \text{Cancel.} \end{aligned}$$

Thus, the equation of the line is $y - y_0 = m(x - x - 0)$.

The Point-Slope form of a line

The equation of the line with slope m that passes through the point (x_0, y_0) is:

$$y - y_0 = m(x - x_0)$$

Example $3.5.1$

Draw the line passing through the point $(-3, -1)$ that has slope $3/5$, then label it with its equation.

Solution

Plot the point $(-3, -1)$, then move 3 units up and 5 units to the right (see Figure $3.5.2$). To find the equation, substitute $(-3, -1)$ for (x_0, y_0) and $3/5$ for m in the point-slope form of the line.

$$\begin{aligned} y - y_0 &= m(x - x_0) \quad \text{Point-slope form.} \\ y - (-1) &= \frac{3}{5}(x - (-3)) \quad \text{Substitute: } 3/5 \text{ for } m, -3 \text{ for } x_0, \text{ and } -1 \text{ for } y_0 \end{aligned}$$

Simplifying, the equation of the line is $y + 1 = \frac{3}{5}(x + 3)$.

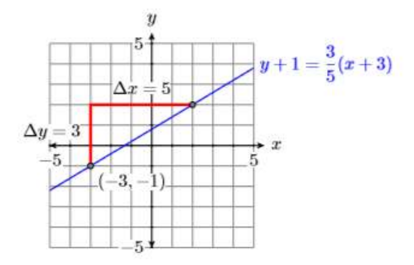
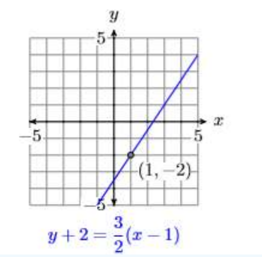


Figure 3.5.2: The line passing through $(-3,-1)$ with slope $3/5$.

Exercise 3.5.1

Draw the line passing through the point $(1,-2)$ that has slope $3/2$, then label it with its equation.

Answer



At this point, you may be asking the question “When should I use the slopintercept form and when should I use the point-slope form?” Here is a good tip.

Tip: Which form should I use

The form you should select depends upon the information given.

1. If you are given the y-intercept and the slope, use $y = mx + b$.
2. If you are given a point and the slope, use $y - y_0 = m(x - x_0)$.

Example 3.5.2

Find the equation of the line passing through the points $P(-1,2)$ and $Q(3,-4)$.

Solution

First, plot the points $P(-1,2)$ and $Q(3,-4)$ and draw a line through them (see Figure 3.5.3).

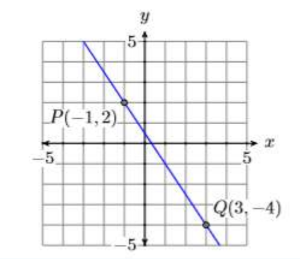


Figure 3.5.3: The line passing through $P(-1,2)$ and $Q(3,-4)$.

Next, let’s calculate the slope of the line by subtracting the coordinates of the point $P(-1,2)$ from the coordinates of point $Q(3,-4)$.

$$\text{Slope} = \frac{\Delta y}{}$$

$$\begin{aligned} &\Delta x \\ &= \frac{-4-2}{3-(-1)} \\ &= \frac{-6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

Thus, the slope is $-3/2$.

Next, use the point-slope form $y - y_0 = m(x - x_0)$ to determine the equation of the line. It's clear that we should substitute $-3/2$ form. But which of the two points should we use? If we use the point $P(-1, 2)$ for (x_0, y_0), we get the answer on the left, but if we use the point $Q(3, -4)$ for (x_0, y_0), we get the answer on the right.

$$y - 2 = -\frac{3}{2}(x - (-1)) \qquad \text{or} \qquad y - (-4) = -\frac{3}{2}(x - 3)$$

At first glance, these answers do not look the same, but let's examine them a bit more closely, solving for y to put each in slope-intercept form. Let's start with the equation on the left.

$$\begin{aligned} y - 2 &= -\frac{3}{2}(x - (-1)) && \text{Using } m = -3/2 \text{ and } (x_0, y_0) = (-1, 2) \\ y - 2 &= -\frac{3}{2}(x + 1) && \text{Simplify.} \\ y - 2 &= -\frac{3}{2}x - \frac{3}{2} && \text{Distribute } -3/2 \\ y - 2 + 2 &= -\frac{3}{2}x - \frac{3}{2} + 2 && \text{Add 2 to both sides.} \\ y &= -\frac{3}{2}x - \frac{3}{2} + \frac{4}{2} && \text{On the left, simplify. On the right, make equivalent} \\ & \text{fractions with a common denominator.} \\ y &= -\frac{3}{2}x + \frac{1}{2} && \text{Simplify.} \end{aligned}$$

Let's put the second equation in slope-intercept form.

> **Note**
>
> Note that the form $y = -\frac{3}{2}x + \frac{1}{2}$, when compared with the general slope-intercept form $y = mx + b$, indicates that the y-intercept is $(0, 1/2)$. Examine Figure 3.5.3. Does it appear that the y-intercept is $(0, 1/2)$?

$$\begin{aligned} y - (-4) &= -\frac{3}{2}(x - 3) && \text{Using } m = -3/2 \text{ and } (x_0, y_0) = (3, -4) \\ y + 4 &= -\frac{3}{2}(x - 3) && \text{Simplify.} \\ y + 4 &= -\frac{3}{2}x + \frac{9}{2} && \text{Distribute } -3/2 \\ y + 4 - 4 &= -\frac{3}{2}x + \frac{9}{2} - 4 && \text{Subtract 4 from both sides.} \\ y &= -\frac{3}{2}x + \frac{9}{2} - \frac{8}{2} && \text{On the left, simplify. On the right, make equivalent} \\ & \text{fractions with a common denominator.} \\ y &= -\frac{3}{2}x + \frac{1}{2} \end{aligned}$$

Thus, both equations simplify to the same answer, $y = -\frac{3}{2}x + \frac{1}{2}$. This means that the equations $y - 2 = -\frac{3}{2}(x - (-1))$ and $y - (-4) = -\frac{3}{2}(x - 3)$, though they look different, are the same.

Exercise 3.5.2

Find the equation of the line passing through the points $P(-2, 1)$ and $Q(4, -1)$.

Answer

$$y = -\frac{1}{3}x + \frac{1}{3}$$

Example 3.5.2 gives rise to the following tip.

tip

When finding the equation of a line through two points P and Q, you may substitute either point P or Q for (x_0, y_0) in the point-slope form $y - y_0 = m(x - x_0)$. The results look different, but they are both equations of the same line.

Parallel Lines

Recall that slope is a number that measures the steepness of the line. If two lines are parallel (never intersect), they have the same steepness.

Parallel lines

If two lines are parallel, they have the same slope.

Example 3.5.3

Sketch the line $y = \frac{3}{4}x - 2$, then sketch the line passing through the point $(-1, 1)$ that is parallel to the line $y = \frac{3}{4}x - 2$. Find the equation of this parallel line.

Solution

Note that $y = \frac{3}{4}x - 2$ is in slope-intercept form $y = mx + b$. Hence,its slope is $3/4$ and its y-intercept is $(0, -2)$. Plot the y-intercept $(0, -2)$, move up 3 units, right 4 units, then draw the line (see Figure 3.5.4).

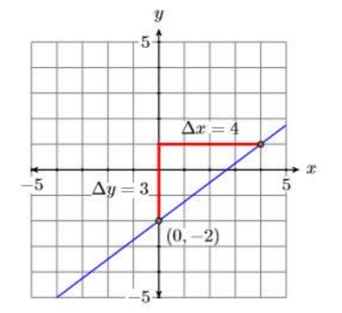


Figure 3.5.4: The line $y = \frac{3}{4}x - 2$.

The second line must be parallel to the first, so it must have the same slope $3/4$. Plot the point $(-1, 1)$, move up 3 units, right 4 units, then draw the line (see the red line in Figure 3.5.5).

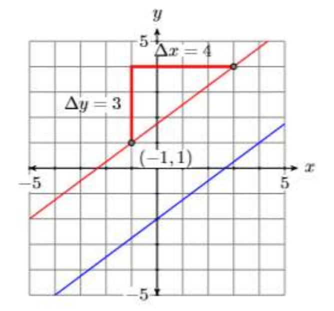


Figure 3.5.5: Adding a line parallel to $y = \frac{3}{4}x - 2$.

To find the equation of the parallel red line in Figure 3.5.5, use the point slope form, substitute $3/4$ for m, then $(-1, 1)$ for (x_0, y_0). That is, substitute -1 for x_0 and 1 for y_0.

$$y - y_0 = m\,(x - x_0) \qquad \text{Point-slope form.}$$

$$y - 1 = \frac{3}{4}(x - (-1)) \qquad \text{Substitute: } 3/4 \text{ for } m, -1 \text{ for } x_0 \text{ and } 1 \text{ for } y_0$$

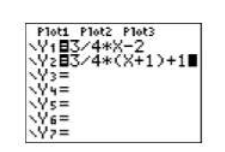
$$y-1=\frac{3}{4}(x+1) \quad \text{Simplify.}$$

Check: In this example, we were not required to solve for y, so we can save ourselves some checking work by writing the equation

$$y-1=\frac{3}{4}(x+1) \qquad \text{in the form} \qquad y=\frac{3}{4}(x+1)+1$$

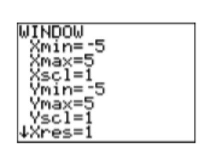
by adding 1 to both sides of the first equation.

Next, enter each equation as shown in Figure $3.5.6$, then change the WINDOW setting as shown in Figure $3.5.7$.



```
Plot1  Plot2  Plot3
\Y1■3/4*X-2
\Y2■3/4*(X+1)+1■
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $3.5.6$: Enter equations of parallel lines.



```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-5
 Ymax=5
 Yscl=1
↓Xres=1
```

Figure $3.5.7$: Adjust the **WINDOW** parameters as shown.

Next, press the GRAPH button, the select **5:ZSquare** to produce the image in Figure $3.5.9$.

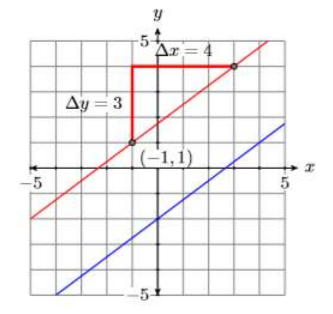


Figure $3.5.8$: Hand-drawn parallel lines.

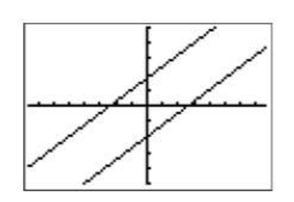


Figure $3.5.9$: Press the GRAPH button then select **5:ZSquare** to produce this image.

Note the close correlation of the calculator lines in Figure $3.5.9$ to the hand drawn lines in Figure $3.5.8$. This gives us confidence that we've captured the correct answer.

Exercise $3.5.3$

Find the equation of the line which passes through the point $(2,-3)$ and is parallel to the line $y=\frac{3}{4}x+2$.

Answer

$$y = \frac{3}{2}x - 6$$

Perpendicular Lines

Two lines are perpendicular if they meet and form a right angle (90 degrees). For example, the lines $\mathscr{L}_1$ and $\mathscr{L}_2$ in Figure $3.5.10$ are perpendicular, but the lines $\mathscr{L}_1$ and $\mathscr{L}_2$ in Figure $3.5.11$ are not perpendicular.

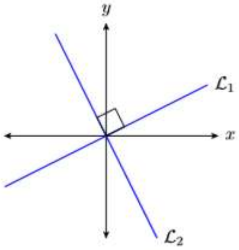


Figure $3.5.10$: The lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are perpendicular. They meet and form a right angle (90 degrees).

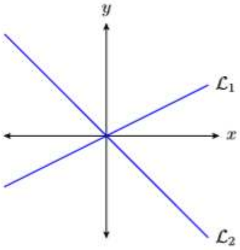


Figure $3.5.11$: The lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are **not** perpendicular. They do **not** form a right angle (90 degrees).

Before continuing, we need to establish a relation between the slopes of two perpendicular lines. So, consider the perpendicular lines $\mathscr{L}_1$ and $\mathscr{L}_2$ in Figure $3.5.12$.

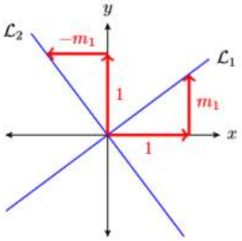


Figure $3.5.12$: Perpendicular lines $\mathscr{L}_1$ and $\mathscr{L}_2$.

Note

1. If we were to rotate line $\mathscr{L}_1$ ninety degrees counter-clockwise, then $\mathscr{L}_1$ would align with the line $\mathscr{L}_2$, as would the right triangles revealing their slopes.
2. $\mathscr{L}_1$ has slope $\frac{\Delta y}{\Delta x} = \frac{m_1}{1} = m_1$.
3. $\mathscr{L}_2$ has slope $\frac{\Delta y}{\Delta x} = \frac{1}{-m_1} = -\frac{1}{m_1}$

Slopes of perpendicular lines

If $\mathscr{L}_1$ and $\mathscr{L}_2$ are perpendicular lines and $\mathscr{L}_1$ has slope m_1, the $\mathscr{L}_2$ has slope $-1/m_1$. That is, the slope of $\mathscr{L}_2$ is the negative reciprocal of the slope of $\mathscr{L}_1$.

Examples:

To find the slope of a perpendicular line, invert the slope of the first line and negate.

- If the slope of $\mathscr{L}_1$ is 2, then the slope of the perpendicular line $\mathscr{L}_2$ is $-1/2$.

- If the slope of $\mathcal{L}_1$ is $-3/4$, then the slope of the perpendicular line $\mathcal{L}_2$ is $4/3$.

Example $3.5.4$

Sketch the line $y=-\frac{2}{3}x-3$, then sketch the line through $(2,1)$ that is perpendicular to the line $y=-\frac{2}{3}x-3$. Find the equation of this perpendicular line.

Solution

Note that $y=-\frac{2}{3}x-3$ is in slope-intercept form $y=mx+b$. Hence, its slope is $-2/3$ and its y-intercept is $(0,-3)$. Plot the y-intercept $(0,-3)$, move right 3 units, down two units, then draw the line (see Figure $3.5.13$).

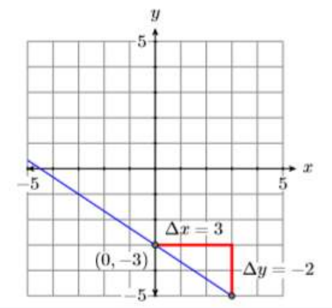


Figure $3.5.13$: The line $y=-\frac{2}{3}x-3$.

Because the line $y=-\frac{2}{3}x-3$ has slope $-2/3$, the slope of the line perpendicular to this line will be the negative reciprocal of $-2/3$, namely $3/2$. Thus, to draw the perpendicular line, start at the given point $(2,1)$, move up 3 units, right 2 units, then draw the line (see Figure $3.5.14$).

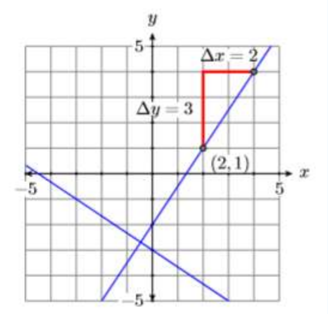


Figure $3.5.14$: Adding a line perpendicular to $y=-\frac{2}{3}x-3$.

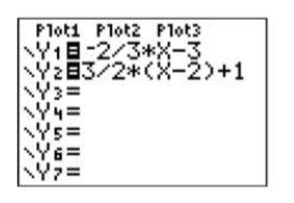
To find the equation of the perpendicular line in Figure $3.5.14$, use the point slope form, substitute $3/2$ for m, then $(2,1)$ for (x_0,y_0). That is, substitute 2 for x_0, then 1 for y_0.

$$\begin{aligned} y-y_0&=m\left(x-x_0\right) && \color{red}{\text{Point-slope form.}} \\ y-1&=\frac{3}{2}(x-2) && \color{red}{\text{Substitute: } 3/2 \text{ for } m, 2 \text{ for } x_0 \text{ and } 1 \text{ for } y_0} \end{aligned}$$

Check: In this example, we were not required to solve for y, so we can save ourselves some checking work by writing the equation

$$y-1=\frac{3}{2}(x-2) \qquad \text{in the form} \qquad y=\frac{3}{2}(x-2)+1$$

by adding 1 to both sides of the first equation. Next, enter each equation as shown in Figure $3.5.15$, then select **6:ZStandard** to produce the image in Figure $3.5.16$.



```
Plot1 Plot2 Plot3
\Y1■-2/3*X-3
\Y2■3/2*(X-2)+1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $3.5.15$: Enter equations of perpendicular lines.

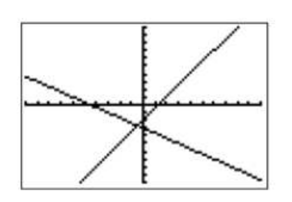
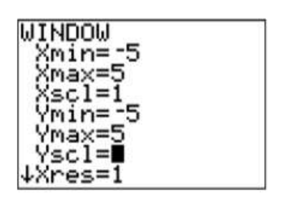


Figure $3.5.16$: **6:ZStandard** produces two lines that do not look perpendicular.

Note that the lines in Figure $3.5.16$ do not appear to be perpendicular. Have we done something wrong? The answer is no! The fact that the calculator's viewing screen is wider than it is tall is distorting the angle at which the lines meet.

To make the calculator result match the result in Figure $3.5.14$, change the window settings as shown in Figure $3.5.17$, then select **5:ZSquare** from the ZOOM menu to produce the image in Figure $3.5.18$. Note the close correlation of the calculator lines in Figure $3.5.18$ to the hand-drawn lines in Figure $3.5.14$. This gives us confidence that we've captured the correct answer.



```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-5
 Ymax=5
 Yscl=■
↓Xres=1
```

Figure $3.5.17$: Change the **WINDOW** parameters as shown.

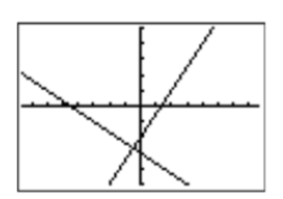


Figure $3.5.18$: **5:ZSquare** produces two lines that do look perpendicular.

Exercise $3.5.4$

Find the equation of the line that passes through the point $(-3,1)$ and is perpendicular to the line $y=-\frac{1}{2}x+1$.

Answer

$y=2x+7$

Applications

Let's look at a real-world application of lines.

Example $3.5.5$

Water freezes at $32^{\circ}\mathrm{F}$ (Fahrenheit) and at $0^{\circ}\mathrm{C}$ (Celsius). Water boils at $212^{\circ}\mathrm{F}$ and at $100^{\circ}\mathrm{C}$. If the relationship is linear, find an equation that expresses the Celsius temperature in terms of the Fahrenheit temperature. Use the result to find the Celsius temperature when the Fahrenheit temperature is $113^{\circ}\mathrm{F}$.

Solution

In this example, the Celsius temperature depends on the Fahrenheit temperature. This makes the Celsius temperature the dependent variable and it gets placed on the vertical axis. This Fahrenheit temperature is the independent variable, so it gets placed on the horizontal axis (see Figure $3.5.19$).

Next, water freezes at 32°F and 0°C, giving us the point $(F, C) = (32, 0)$. Secondly, water boils at 212°F and 100°C, giving us the point $(F, C) = (212, 100)$. Note how we've scaled the axes so that each of these points fit on the coordinate system. Finally, assuming a linear relationship between the Celsius and Fahrenheit temperatures, draw a line through these two points (see Figure $3.5.19$).

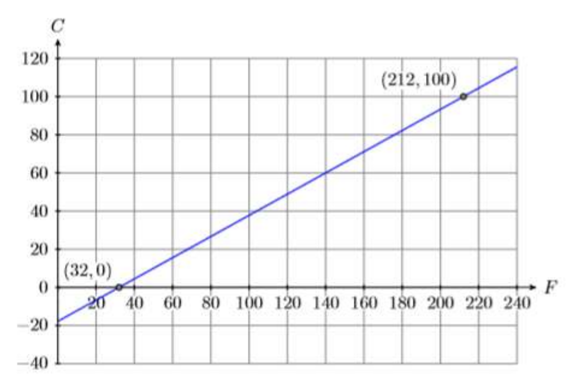


Figure $3.5.19$: The linear relationship between Celsius and Fahrenheit temperature.

Calculate the slope of the line.

$$m = \frac{\Delta C}{\Delta F} \quad \text{Slope formula.}$$
$$m = \frac{100 - 0}{212 - 32} \quad \text{Use the points } (32, 0) \text{ and } (212, 100) \text{ to compute the difference in C and F}$$
$$m = \frac{100}{180} \quad \text{Simplify.}$$
$$m = \frac{5}{9} \quad \text{Reduce.}$$

You may either use $(32, 0)$ or $(212, 100)$ in the point-slope form. The point $(32, 0)$ has smaller numbers, so it seems easier to substitute $(x_0, y_0) = (32, 0)$ and $m = 5/9$ into the point-slope form $y - y_0 = m(x - x_0)$.

$$y - y_0 = m(x - x_0) \quad \text{Point-slope form.}$$
$$y - 0 = \frac{5}{9}(x - 32) \quad \text{Substitute: } 5/9 \text{ for } m, 32 \text{ for } x_0, \text{ and } 0 \text{ for } y_0$$
$$y = \frac{5}{9}(x - 32) \quad \text{Simplify.}$$

However, our vertical and horizontal axes are labeled C and F (see Figure \(\PageIndex{19}\)) respectively, so we must replace y with C and x with F to obtain an equation expressing the Celsius temperature C in terms of the Fahrenheit temperature F.

$$C=\frac{5}{9}(F-32) \tag{3.5.1}$$

Finally, to find the Celsius temperature when the Fahrenheit temperature is $113^{\circ}\mathrm{F}$, substitute 113 for F in Equation 3.5.1

$$\begin{aligned} C &= \frac{5}{9}(F-32) \quad \color{red}{\text{Equation (3.5.1)}} \\ C &= \frac{5}{9}(113-32) \quad \color{red}{\text{Substitute: } 113 \text{ for } F} \\ C &= \frac{5}{9}(81) \quad \color{red}{\text{Subtract.}} \\ C &= 45 \quad \color{red}{\text{Multiply.}} \end{aligned}$$

Therefore, if the Fahrenheit temperature is $113^{\circ}\mathrm{F}$, then the Celsius temperature is $45^{\circ}\mathrm{C}$.

Exercise 3.5.5

Find an equation that expresses the Fahrenheit temperature in terms of the Celsius temperature. Use the result to find the Fahrenheit temperature when the Celsius temperature is $25^{\circ}\mathrm{C}$.

Answer

$77^{\circ}\mathrm{F}$

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3.6: Standard Form of a Line

In this section we will investigate the standard form of a line. Let's begin with a simple example.

Example 3.6.1

Solve the equation $2x+3y=6$ for y and plot the result.

Solution

First we solve the equation $2x+3y=6$ for y. Begin by isolating all terms containing y on one side of the equation, moving or keeping all the remaining terms on the other side of the equation.

$$\begin{aligned} 2x+3y&=6 && \text{Original equation.} \\ 2x+3y-2x&=6-2x && \text{Subtract } 2x \text{ from both sides.} \\ 3y&=6-2x && \text{Simplify.} \\ \frac{3y}{3}&=\frac{6-2x}{3} && \text{Divide both sides by 3} \end{aligned}$$

Note

Just as multiplication is distributive with respect to addition

$$a(b+c)=ab+ac$$

so too is division distributive with respect to addition.

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

When dividing a sum or a difference by a number, we use the distributive property and divide both terms by that number.

$$\begin{aligned} y&=\frac{6}{3}-\frac{2x}{3} && \text{On the left, simplify. On the right, divide both terms by 3} \\ y&=2-\frac{2x}{3} && \text{Simplify.} \end{aligned}$$

Finally, use the commutative property to switch the order of the terms on the right-hand side of the last result.

$$\begin{aligned} y&=2+\left(-\frac{2x}{3}\right) && \text{Add the opposite.} \\ y&=-\frac{2}{3}x+2 && \text{Use the commutative property to switch the order.} \end{aligned}$$

Because the equation $2x+3y=6$ is equivalent to the equation $y=-\frac{2}{3}x+2$, the graph of $2x+3y=6$ is a line, having slope $m=-2/3$ and y-intercept $(0,2)$. Therefore, to draw the graph of $2x+3y=6$, plot they-intercept $(0,2)$, move down 2 and 3 to the right, then draw the line (see Figure $3.6.1$).

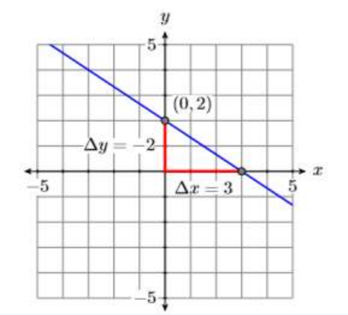
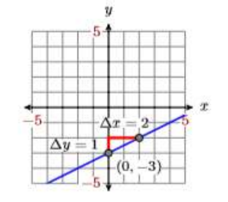


Figure $3.6.1$: The graph of $2x+3y=6$, or equivalently, $y=-\frac{2}{3}x+2$

Exercise 3.6.1

Add exercises text here.

Answer



In general, unless $B = 0$, we can always solve the equation $Ax + By = C$ for y:

$$\begin{aligned} Ax + By &= C && \text{Original equation.} \\ Ax + By - Ax &= C - Ax && \text{Subtract } Ax \text{ from both sides.} \\ By &= C - Ax && \text{Simplify.} \\ \frac{By}{B} &= \frac{C - Ax}{B} && \text{Divide both sides by } B \\ y &= \frac{C}{B} - \frac{Ax}{B} && \text{distribute the } B \\ y &= -\frac{A}{B}x + \frac{C}{B} && \text{Commutative property} \end{aligned}$$

Note that the last result is in slope-intercept form $y = mx + b$, whose graph is a line. We have established the following result.

FAct

The graph of the equation $Ax + By = C$, is a line.

Important points: A couple of important comments are in order.

1. The form $Ax + By = C$ requires that the coefficients A, B, and C are integers. So, for example, we would clear the fractions from the form

   $$\frac{1}{2}x + \frac{2}{3}y = \frac{1}{4}$$

   by multiplying both sides by the least common denominator.

   $$\begin{aligned} 12\left(\frac{1}{2}x + \frac{2}{3}y\right) &= \left(\frac{1}{4}\right)12 \\ 6x + 8y &= 3 \end{aligned}$$

   Note that the coefficients are now integers.
2. The form $Ax + By = C$ also requires that the first coefficient A is nonnegative; i.e., $A \geq 0$. Thus, if we have

   $$-5x + 2y = 6$$

   then we would multiply both sides by -1, arriving at:

   $$\begin{aligned} -1(-5x + 2y) &= (6)(-1) \\ 5x - 2y &= -6 \end{aligned}$$

   Note that $A = 5$ is now greater than or equal to zero.
3. If A, B, and C have a common divisor greater than 1, it is recommended that we divide both sides by the common divisor, thus “reducing” the coefficients. For example, if we have

   $$3x + 12y = -24$$

   then dividing both side by 3 “reduces” the size of the coefficients.

   $$\frac{3x + 12y}{3} = \frac{-24}{3}$$

$$x+4y=-8$$

Standard form

The form $Ax+By=C$, where A, B, and C are integers, and $A \geq 0$, is called the standard form of a line.

Slope-Intercept to Standard Form

We've already transformed a couple of equations in standard form into slopeintercept form. Let's reverse the process and place an equation in slope intercept form into standard form.

Example $3.6.2$

Given the graph of the line in Figure $3.6.2$, find the equation Given the graph of the line below, find the equation of the line in standard form.

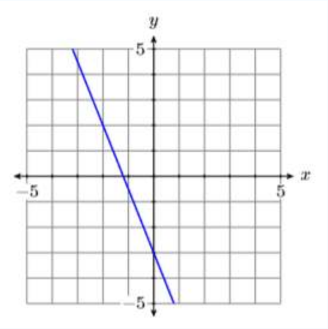


Figure $3.6.2$: Determine the equation of the line.

Solution

The line intercepts the y-axis at $(0,-3)$. From $(0,-3)$, move up 5 units, then left 2 units. Thus, the line has slope $\Delta y/\Delta x=-5/2$ (see Figure $3.6.3$). Substitute $-5/2$ form and -3 for b in the slope-intercept form of the line.

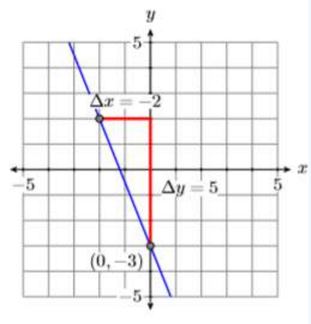


Figure $3.6.3$: The line has y-intercept $(0,-3)$ and slope $-5/2$.

$$\begin{aligned} y&=mx+b && \text{Slope-intercept form.} \\ y&=-\frac{5}{2}x-3 && \text{Substitute: } -5/2 \text{ for } m, -3 \text{ for } b \end{aligned}$$

To put this result in standard form $Ax+By=C$, first clear the fractions by multiplying both sides by the common denominator.

$$\begin{aligned} 2y&=2\left[-\frac{5}{2}x-3\right] && \text{Multiply both sides by 2} \\ 2y&=2\left[-\frac{5}{2}x\right]-2[3] && \text{Distribute the 2} \\ 2y&=-5x-6 && \text{Multiply.} \end{aligned}$$

That clears the fractions. To put this last result in the form $Ax+By=C$, we need to move the term $-5x$ to the other side of the equation.

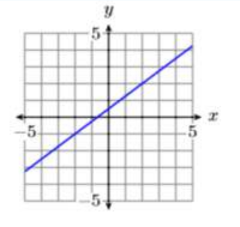
$$5x+2y=-5x-6+5x \qquad \text{Add } 5x \text{ to both sides.}$$

$$5x+2y=-6 \quad \text{Simplify.}$$

Thus, the standard form of the line is $5x+2y=-6$. Note that all the coefficients are integers and the terms are arranged in the order $Ax+By=C$, with $A \geq 0$.

Exercise 3.6.2

Given the graph of the line below, find the equation of the line in standard form.



Answer

$3x-4y=-2$

Point-Slope to Standard Form

Let’s do an example where we have to put the point-slope form of a line in standard form.

Example 3.6.3

Sketch the line passing through the points $(-3,-4)$ and $(1,2)$, then find the equation of the line in standard form.

Solution

Plot the points (−3,−4) and (1,2), then draw a line through them (see Figure 3.6.4).

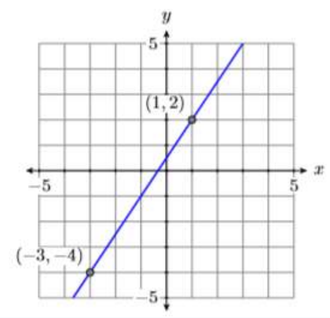


Figure 3.6.4: The line through $(-3,-4)$ and $(1,2)$.

Use the points $(-3,-4)$ and $(1,2)$ to calculate the slope.

$$\begin{aligned}
\text{Slope } &= \frac{\Delta y}{\Delta x} && \text{Slope formula.}\\
&= \frac{2-(-4)}{1-(-3)} && \text{Subtract coordinates of } (-3,-4)\\
&= \frac{6}{4} && \text{Simplify.}\\
&= \frac{3}{2} && \text{Reduce.}
\end{aligned}$$

Let’s substitute $(x_0,y_0)=(1,2)$ and $m=3/2$ in the point-slope form of the line. (*Note: Substituting* $(x_0,y_0)=(-3,-4)$ *and* $m=3/2$ *would yield the same answer.*)

$$\begin{aligned}
y-y_0 &= m\left(x-x_0\right) && \text{Point-slope form.}\\
y-2 &= \frac{3}{2}(x-1) && \text{Substitute: } 3/2 \text{ for } m, 1 \text{ for } x_0
\end{aligned}$$

The question requests that our final answer be presented in standard form. First we clear the fractions.

$$\begin{aligned} y-2 &= \frac{3}{2}x - \frac{3}{2} && \text{Distribute the } 3/2 \\ 2[y-2] &= 2\left[\frac{3}{2}x - \frac{3}{2}\right] && \text{Multiply both sides by } 2 \\ 2y - 2[2] &= 2\left[\frac{3}{2}x\right] - 2\left[\frac{3}{2}\right] && \text{Distribute the } 2 \\ 2y-4 &= 3x-3 && \text{Multiply.} \end{aligned}$$

Now that we've cleared the fractions, we must order the terms in the form $Ax + By = C$. We need to move the term $3x$ to the other side of the equation.

$$\begin{aligned} 2y-4-3x &= 3x-3-3x && \text{Subtract } 3x \text{ from both sides.} \\ -3x+2y-4 &= -3 && \text{Simplify, changing the order on the left-hand side.} \end{aligned}$$

To put this in the form $Ax + By = C$, we need to move the term -4 to the other side of the equation.

$$\begin{aligned} -3x+2y-4+4 &= -3+4 && \text{Add 4 to both sides.} \\ -3x+2y &= 1 && \text{Simplify.} \end{aligned}$$

It appears that $-3x+2y=1$ is in the form $Ax+By=C$. However, standard form requires that $A \geq 0$. We have $A=-3$. To fix this, we multiply both sides by -1.

$$\begin{aligned} -1[-3x+2y] &= -1[1] && \text{Multiply both sides by } -1 \\ 3x-2y &= -1 && \text{Distribute the } -1 \end{aligned}$$

Thus, the equation of the line in standard form is $3x-2y=-1$.

Note

If we fail to reduce the slope to lowest terms, then the equation of the line would be:

$$y-2=\frac{6}{4}(x-1)$$

Multiplying both sides by 4 would give us the result

$$4y-8=6x-6$$

or equivalently:

$$-6x+4y=2$$

This doesn't look like the same answer, but if we divide both sides by -2, we do get the same result.

$$3x-2y=-1$$

This shows the importance of requiring $A \geq 0$ and "reducing" the coefficients A, B, and C. It allows us to compare our answer with our colleagues or the answers presented in this textbook.

Exercise 3.6.3

Find the standard form of the equation of the line that passes through the points $(-2,4)$ and $(3,-3)$.

Answer

$7x+5y=6$

Intercepts

We've studied the y-intercept, the point where the graph crosses the y-axis, but equally important are the x-intercepts, the points where the graph crosses the x-axis.

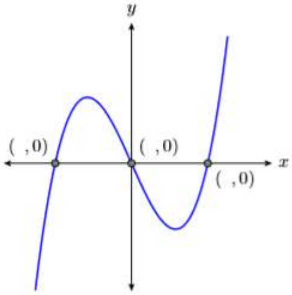


Figure 3.6.5: Each x-intercept has a y-coordinate equal to zero.

In Figure 3.6.5, the graph crosses the x-axis three times. Each of these crossing points is called an x-intercept. Note that each of these x-intercepts has a y-coordinate equal to zero. This leads to the following rule.

x Intercepts

To find the x-intercepts of the graph of an equation, substitute $y=0$ into the equation and solve for x.

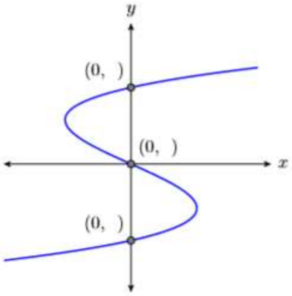


Figure 3.6.6: Each y-intercept has an x-coordinate equal to zero.

Similarly, the graph in Figure 3.6.6 crosses the y-axis three times. Each of these crossing points is called a y-intercept. Note that each of these y-intercepts has an x-coordinate equal to zero. This leads to the following rule.

y Intercepts

To find the y-intercepts of the graph of an equation, substitute $x=0$ into the equation and solve for y.

Let’s put these rules for finding intercepts to work.

Example 3.6.4

Find the x- and y-intercepts of the line having equation $2x-3y=6$. Plot the intercepts and draw the line.

Solution

We know that the graph of $2x-3y=6$ is a line. Furthermore, two points completely determine a line. This means that we need only plot the x- and y-intercepts, then draw a line through them.

To find the x-intercept of $2x-3y=6$, substitute 0 for y and solve for x.

$$\begin{aligned} 2x-3y&=6 \\ 2x-3(0)&=6 \\ 2x&=6 \\ \frac{2x}{2}&=\frac{6}{2} \\ x&=3 \end{aligned}$$

Thus, the x-intercept of the line is $(3,0)$.

To find the y-intercept of $2x-3y=6$, substitute 0 for x and solve for y.

$$2x-3y=6$$

$$\begin{aligned} 2(0)-3y&=6\\ -3y&=6\\ \frac{-3y}{-3}&=\frac{6}{-3}\\ y&=-2 \end{aligned}$$

Thus, the y-intercept of the line is $(0,-2)$.

Plot the x-intercept $(3,0)$ and the y-intercept $(0,-2)$ and draw a line through them (see Figure $3.6.7$).

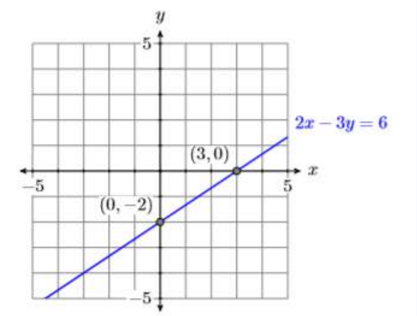


Figure $3.6.7$: The graph of $2x-3y=6$ has intercepts $(3,0)$ and $(0,-2)$.

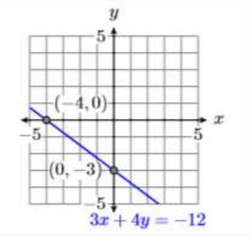
Exercise $3.6.4$

Find the x- and y-intercepts of the line having equation $3x+4y=-12$. Plot the intercepts and draw the line.

Answer

x-intercept: $(-4,0)$

y-intercept: $(0,-3)$



Example $3.6.5$

Sketch the line $4x+3y=12$, then sketch the line through the point $(-2,-2)$ that is perpendicular to the line $4x+3y=12$. Find the equation of this perpendicular line.

Solution

Let's first find the x- and y-intercepts of the line $4x+3y=12$.

To find the x-intercept of the line $4x+3y=12$, substitute 0 for y and solve for x.

$$\begin{aligned} 4x+3y&=12\\ 4x+3(0)&=12\\ 4x&=12\\ \frac{4x}{4}&=\frac{12}{4}\\ x&=3 \end{aligned}$$

Thus, the x-intercept of the line is $(3,0)$.

To find the y-intercept of the line $4x+3y=12$, substitute 0 for x and solve for y.

$$\begin{aligned} 4x+3y&=12\\ 4(0)+3y&=12\\ 3y&=12\\ \frac{3y}{3}&=\frac{12}{3}\\ y&=4 \end{aligned}$$

Thus, the y-intercept of the line is $(0,4)$.

Plot the intercepts and draw a line through them. Note that it is clear from the graph that the slope of the line $3x+4y=12$ is $-4/3$ (see Figure $3.6.8$).

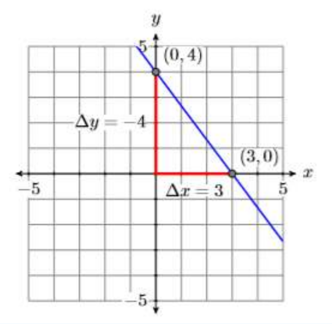


Figure $3.6.8$: The graph of $4x+3y=12$ has intercepts $(3,0)$ and $(0,4)$ and slope $-4/3$.

You could also solve for y to put $3x+4y=12$ in slope intercept form in order to determine the slope.

Because the slope of $3x+4y=12$ is $-4/3$, the slope of a line perpendicular to $3x+4y=12$ will be the negative reciprocal of $-4/3$, namely $3/4$. Our perpendicular line has to pass through the point $(-2,-2)$. Start at $(-2,-2)$, move 3 units upward, then 4 units to the right, then draw the line. It should appear to be perpendicular to the line $3x+4y=12$ (see Figure $3.6.9$).

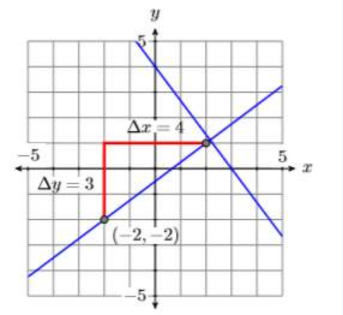


Figure $3.6.9$: The slope of the perpendicular line is the negative reciprocal of $-4/3$, namely $3/4$.

Finally, use the point-slope form, $m=3/4$, and $(x_0,y_0)=(-2,-2)$ to determine the equation of the perpendicular line.

$$\begin{aligned} y-y_0&=m\left(x-x_0\right) && \text{Point-slope form.}\\ y-(-2)&=\frac{3}{4}(x-(-2)) && \text{Substitute: } 3/4 \text{ for } m, -2 \text{ for } x_0 \text{ and } -2 \text{ for } y_0\\ y+2&=\frac{3}{4}(x+2) && \text{Simplify.} \end{aligned}$$

Let’s place our answer in standard form. Clear the fractions.

$$\begin{aligned} y+2&=\frac{3}{4}x+\frac{6}{4} && \text{Distribute } 3/4\\ 4[y+2]&=4\left[\frac{3}{4}x+\frac{6}{4}\right] && \text{Multiply both sides by } 4\\ 4y+4[2]&=4\left[\frac{3}{4}x\right]+4\left[\frac{6}{4}\right] && \text{Distribute the } 4\\ 4y+8&=3x+6 && \text{Multiply.} \end{aligned}$$

Rearrange the terms to put them in the order $Ax+By=C$.

$$\begin{aligned} 4y+8-3x &= 3x+6-3x && \text{Subtract } 3x \text{ from both sides.} \\ -3x+4y+8 &= 6 && \text{Simplify. Rearrange on the left.} \\ -3x+4y+8-8 &= 6-8 && \text{Subtract } 8 \text{ from both sides.} \\ -3x+4y &= -2 && \text{Simplify.} \\ -1(-3x+4y) &= -1(-2) && \text{Multiply both sides by } -1 \\ 3x-4y &= 2 && \text{Distribute the } -1 \end{aligned}$$

Hence, the equation of the perpendicular line is $3x-4y=2$.

Exercise $3.6.5$

Find the equation of the line that passes through the point $(3,2)$ and is perpendicular to the line $6x-5y=15$.

Answer

$5x+6y=27$

Horizontal and Vertical Lines

Here we keep an earlier promise to address what happens to the standard form $Ax+By=C$ when either $A=0$ or $B=0$. For example, the form $3x=6$, when compared with the standard form $Ax+By=C$, has $B=0$. Similarly, the form $2y=-12$, when compared with the standard form $Ax+By=C$, has $A=0$. Of course, $3x=6$ can be simplified to $x=2$ and $2y=-12$ can be simplified to $y=-6$. Thus, if either $A=0$ or $B=0$, the standard form Ax + By = C takes the form $x=a$ and $y=b$, respectively.

As we will see in the next example, the form $x=a$ produces a vertical line, while the form $y=b$ produces a horizontal line.

Example $3.6.6$

Sketch the graphs of $x=3$ and $y=-3$.

Solution

To sketch the graph of $x=3$, recall that the graph of an equation is the set of all points that satisfy the equation. Hence, to draw the graph of $x=3$, we must plot all of the points that satisfy the equation $x=3$; that is, we must plot all of the points that have an x-coordinate equal to 3. The result is shown in Figure $3.6.10$.

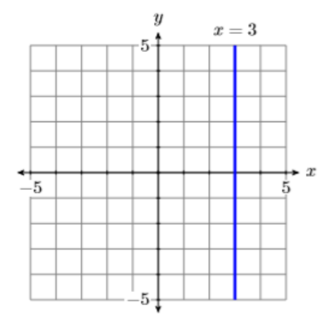


Figure $3.6.10$: The graph of $x=3$ is a vertical line.

Secondly, to sketch the graph of $y=-3$, we plot all points having a y-coordinate equal to -3. The result is shown in Figure $3.6.11$.

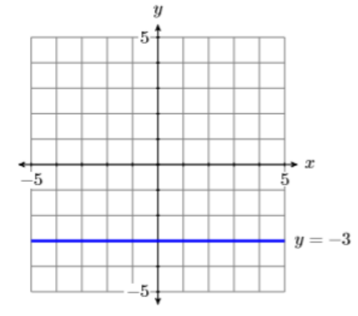


Figure 3.6.11: The graph of $y=-3$ is a horizontal line.

Things to note:

A couple of comments are in order regarding the lines in Figures 3.6.10 and 3.6.11.

1. The graph of $x=3$ in Figure 3.6.10, being a vertical line, has undefined slope. Therefore, we cannot use either of the formulae $y=mx+b$ or $y-y_0=m(x-x_0)$ to obtain the equation of the line. The only way we can obtain the equation is to note that the line is the set of all points (x,y) whose x-coordinate equals 3.
2. However, the graph of $y=-3$, being a horizontal line, has slope zero, so we can use the slope-intercept form to find the equation of the line. Note that the y-intercept of this graph is $(0,-3)$. If we substitute these numbers into $y=mx+b$, we get:

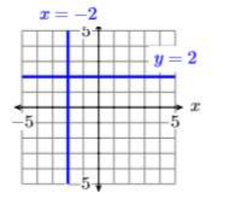
$$\begin{aligned} y &= mx+b && \text{Slope-intercept form.} \\ y &= 0x+(-3) && \text{Substitute: 0 for } m, -3 \text{ for } b \\ y &= -3 && \text{Simplify.} \end{aligned}$$

However, it is far easier to just look at the line in Figures 3.6.11 and note that it is the collection of all points (x,y) with $y=3$.

Exercise 3.6.6

Sketch the graphs of $x=-2$ and $y=2$.

Answer



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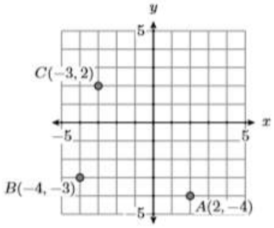
3.E: Introduction to Graphing (Exercises)

3.1: Graphing Equations by Hand

In Exercises 1-6, set up a Cartesian Coordinate system on a sheet of graph paper, then plot the given points.

1) $A(2,-4), B(-4,-3)$, and $C(-3,2)$

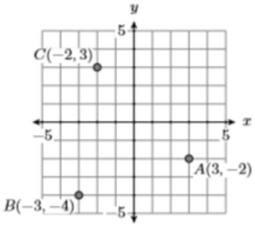
Answer



2) $A(3,-4), B(-3,-2)$, and $C(-4,4)$

3) $A(3,-2), B(-3,-4)$, and $C(-2,3)$

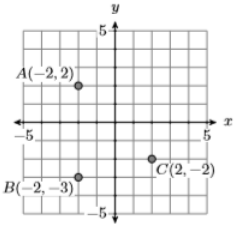
Answer



4) $A(3,-4), B(-3,-4)$, and $C(-4,2)$

5) $A(-2,2), B(-2,-3)$, and $C(2,-2)$

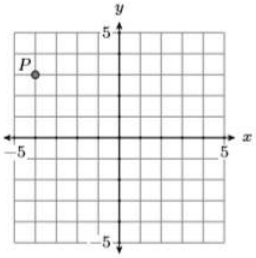
Answer



6) $A(-3,2), B(2,-4)$, and $C(-3,-4)$

In Exercises 7-10, identify the coordinates of point P in each of the given coordinate systems.

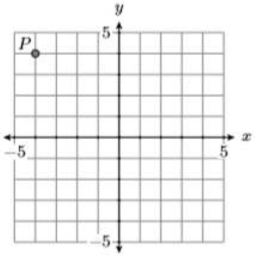
7)



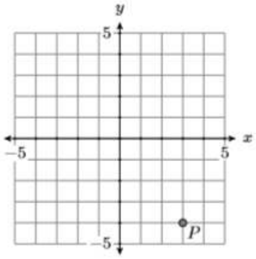
Answer

$(-4, 3)$

8)



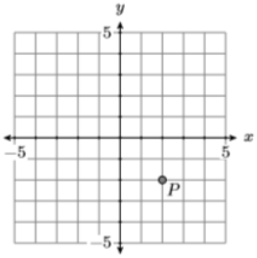
9)



Answer

$(3, -4)$

10)



11) Which of the points $(8,36)$, $(5,20)$, $(4,13)$, and $(-2,-12)$ is a solution of the equation $y=5x-5$?

Answer

$(5,20)$

12) Which of the points $(0,-9)$, $(6,-46)$, $(1,-14)$, and $(-6,36)$ is a solution of the equation $y=-7x-7$?

13) Which of the points $(-9,49)$, $(1,1)$, $(-6,39)$, and $(2,-3)$ is a solution of the equation $y=-5x+6$?

Answer

$(1,1)$

14) Which of the points $(7,-15)$, $(-9,30)$, $(-6,19)$, and $(5,-11)$ is a solution of the equation $y=-3x+3$?

15) Which of the points $(1,12)$, $(7,395)$, $(-7,398)$, and $(0,1)$ is a solution of the equation $y=8x^2+3$?

Answer

$(7,395)$

16) Which of the points $(-7,154)$, $(7,153)$, $(-2,21)$, and $(2,16)$ is a solution of the equation $y=3x^2+6$?

17) Which of the points $(8,400)$, $(5,158)$, $(0,3)$, and $(2,29)$ is a solution of the equation $y=6x^2+2x$?

Answer

$(8,400)$

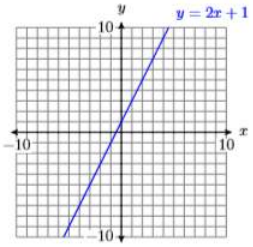
18) Which of the points $(-6,-114)$, $(6,-29)$, $(-7,-149)$, and $(4,-1)$ is a solution of the equation $y=-2x^2+7x$?

In Exercises 19-26, complete the table of points that satisfy the given equation. Set up a coordinate system on a sheet of graph paper, plot the points from the completed table, then draw the graph of the given equation.

19)

x	$y=2x+1$	(x,y)
-3		
-2		
-1		
0		
1		
2		
3		

Answer



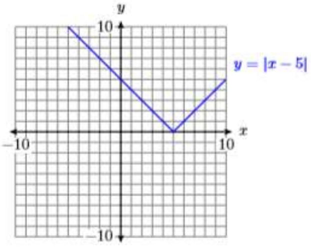
20)

x	$y=x-4$	(x,y)
-3		
-2		
-1		
0		
1		
2		
3		

21)

x	$y=\|x-5\|$	(x,y)
2		
3		
4		
5		
6		
7		
8		

Answer



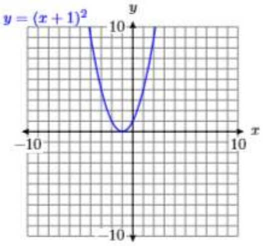
22)

x	$y=\|x+2\|$	(x,y)
-5		
-4		
-3		
-2		
-1		
0		
1		

23)

x	$y=(x+1)^2$	(x,y)
-4		
-3		
-2		
-1		
0		
1		
2		

Answer



24)

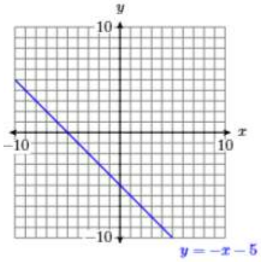
x	$y=(x+5)^2$	(x,y)
-8		
-7		
-6		
-5		
-4		
-3		
-2		

25)

x	$y=-x-5$	(x,y)
-3		
-2		
-1		
0		
1		
2		

x	$y=-x-5$	(x, y)
3		

Answer

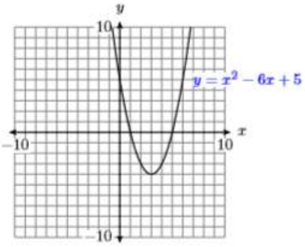


26)

x	$y=-2x+3$	(x, y)
-3		
-2		
-1		
0		
1		
2		
3		

27) Use a graphing calculator to complete a table of points satisfying the equation $y=x^2-6x+5$. Use integer values of x, starting at 0 and ending at 6. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y=x^2-6x+5$.

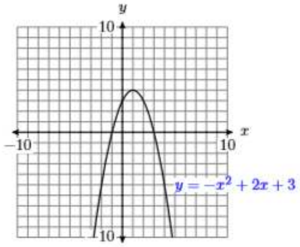
Answer



28) Use a graphing calculator to complete a table of points satisfying the equation $y=x^2-2x-3$. Use integer values of x, starting at $-2$and ending at 4. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y=x^2-2x-3$.

29) Use a graphing calculator to complete a table of points satisfying the equation $y=-x^2+2x+3$. Use integer values of x, starting at -2 and ending at 4. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y=-x^2+2x+3$.

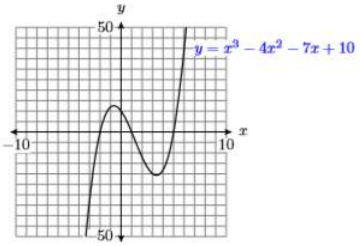
Answer



30) Use a graphing calculator to complete a table of points satisfying the equation $y = -x^2 - 2x + 8$. Use integer values of x, starting at -5 and ending at 3. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = -x^2 - 2x + 8$.

31) Use a graphing calculator to complete a table of points satisfying the equation $y = x^3 - 4x^2 - 7x + 10$. Use integer values of x, starting at -3 and ending at 6. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = x^3 - 4x^2 - 7x + 10$.

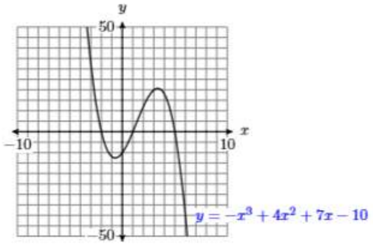
Answer



32) Use a graphing calculator to complete a table of points satisfying the equation $y = x^3 + 3x^2 - 13x - 15$. Use integer values of x, starting at -6 and ending at 4. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = x^3 + 3x^2 - 13x - 15$.

33) Use a graphing calculator to complete a table of points satisfying the equation $y = -x^3 + 4x^2 + 7x - 10$. Use integer values of x, starting at -3 and ending at 6. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = -x^3 + 4x^2 + 7x - 10$.

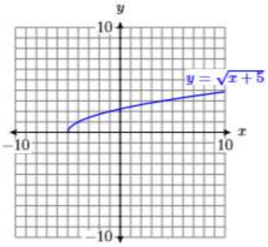
Answer



34) Use a graphing calculator to complete a table of points satisfying the equation $y = -x^3 + x^2 + 12x$. Use integer values of x, starting at -4 and ending at 5. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = -x^3 + x^2 + 12x$.

35) Use a graphing calculator to complete a table of points satisfying the equation $y = \sqrt{x + 5}$. Use integer values of x, starting at -5 and ending at 10. Round your y-values to the nearest tenth. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y = \sqrt{x + 5}$.

Answer



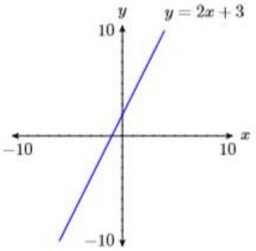
36) Use a graphing calculator to complete a table of points satisfying the equation $y=\sqrt{4-x}$. Use integer values of x, starting at -10 and ending at 4. Round your y-values to the nearest tenth. After completing the table, set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot the points in your table. Finally, use the plotted points as evidence to draw the graph of $y=\sqrt{4-x}$.

3.2: The Graphing Calculator

In Exercises 1-16, enter the given equation in the Y= menu of your calculator, then select 6:ZStandard from the ZOOM menu to produce its graph. Follow the Calculator Submission Guidelines outlined in the subsection Reproducing Calculator Results on Homework Paper when submitting your result on your homework paper.

1) $y=2x+3$

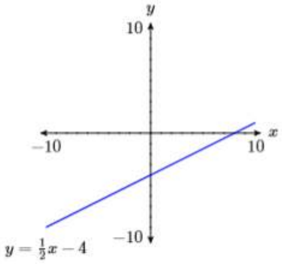
Answer



2) $y=3-2x$

3) $y=\frac{1}{2}x-4$

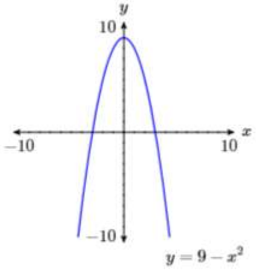
Answer



4) $y=-\frac{2}{3}x+2$

5) $y=9-x^2$

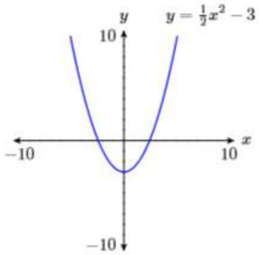
Answer



6) $y = x^2 - 4$

7) $y = \frac{1}{2}x^2 - 3$

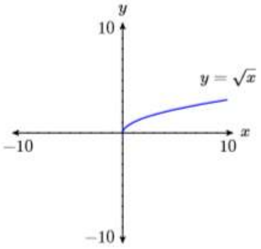
Answer



8) $y = 4 - \frac{1}{3}x^2$

9) $y = \sqrt{x}$

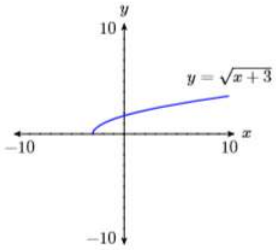
Answer



10) $y = \sqrt{-x}$

11) $y = \sqrt{x+3}$

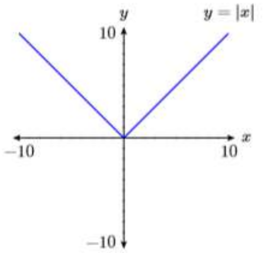
Answer



12) $y = \sqrt{5-x}$

13) $y=|x|$

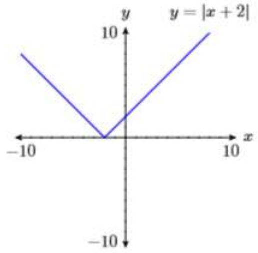
Answer



14) $y=-|x|$

15) $y=|x+2|$

Answer

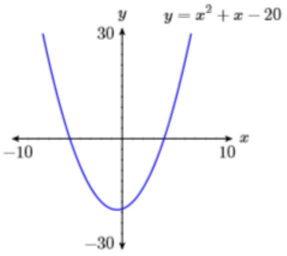


16) $y=-|x-3|$

In Exercises 17-22, the graph of each given equation is a parabola, having a "u-shape" similar to the graph in Figure 3.2.13 Some of the parabolas "open up" and some "open down." Adjust Ymin and/or Ymax in the WINDOW menu so that the "turning point" of the parabola, called its vertex, is visible in the viewing screen of your graphing calculator. Follow the *Calculator Submission Guidelines* outlined in the subsection Reproducing Calculator Results on Homework Paper when submitting your result on your homework paper.

17) $y=x^2+x-20$

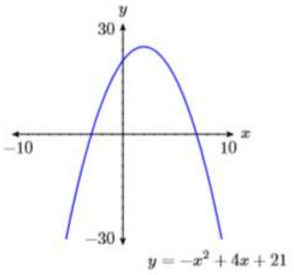
Answer



18) $y=-x^2-2x+24$

19) $y=-x^2+4x+21$

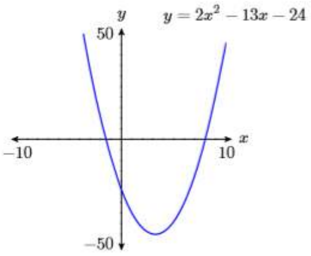
Answer



20) $y=x^2+6x-16$

21) $y=2x^2-13x-24$

Answer

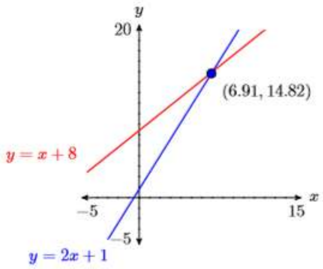


22) $y=-3x^2-19x+14$

In Exercises 23-28, sketch the given lines, then adjust the WINDOW parameters so that the point of intersection of the two lines is visible in the viewing window. Use the TRACE key to estimate the point of intersection. Follow the *Calculator Submission Guidelines* outlined in the subsection Reproducing Calculator Results on Homework Paper when submitting your result on your homework paper. Be sure to also include your estimate of the point of intersection in your sketch.

23) $y=2x+1$ and $y=x+8$

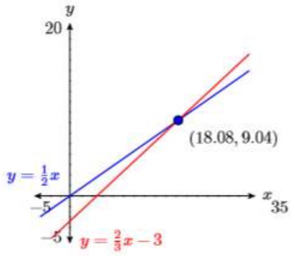
Answer



24) $y=4-2x$ and $y=-6-x$

25) $y=\frac{1}{2}x$ and $y=\frac{2}{3}x-3$

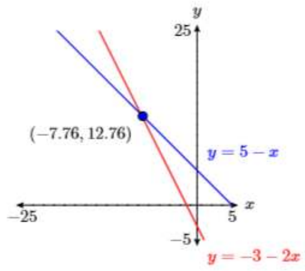
Answer



26) $y=2x+5$ and $y=x-6$

27) $y=5-x$ and $y=-3-2x$

Answer



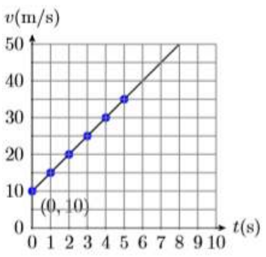
28) $y=1-\dfrac{1}{2}x$ and $y=-4-\dfrac{4}{5}x$

3.3: Rates and Slope

1) An object's initial velocity at time $t=0$ seconds is $v=10$ meters per second. It then begins to pick up speed (accelerate) at a rate of 5 meters per second per second (5m/s/s or 5m/s^2).

a. Set up a Cartesian Coordinate System on a sheet of graph paper. Label and scale each axis. Include units with your labels.
b. Plot the point representing the initial velocity at time $t=0$ seconds. Then plot a minimum of 5 additional points using the fact that the object is accelerating at a rate of 5 meters per second per second.
c. Sketch the line representing the object's velocity versus time.
d. Calculate the slope of the line.

Answer



2) An object's initial velocity at time $t=0$ seconds is $v=40$ meters per second. It then begins to lose speed at a rate of 5 meters per second per second (5m/s/s or 5m/s^2).

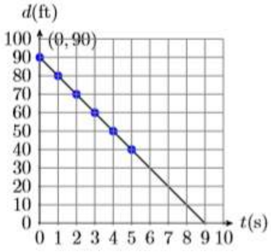
a. Set up a Cartesian Coordinate System on a sheet of graph paper. Label and scale each axis. Include units with your labels.
b. Plot the point representing the initial velocity at time $t=0$ seconds. Then plot a minimum of 5 additional points using the fact that the object is losing speed at a rate of 5 meters per second per second.
c. Sketch the line representing the object's velocity versus time.

d. Calculate the slope of the line.

3) David first sees his brother when the distance separating them is 90 feet. He begins to run toward his brother, decreasing the distance d between him and his brother at a constant rate of 10 feet per second (10 ft/s).

a. Set up a Cartesian Coordinate System on a sheet of graph paper. Label and scale each axis. Include units with your labels.
b. Plot the point representing David's initial distance from his brother at time t = 0 seconds. Then plot a minimum of 5 additional points using the fact that David's distance from his brother is decreasing at a constant rate of 10 feet per second ($10\mathrm{ft}/\mathrm{s}$).
c. Sketch the line representing David's distance from his brother versus time.
d. Find the slope of the line.

Answer:



4) David initially stands 20 feet from his brother when he sees his girl friend Mary in the distance. He begins to run away from his brother and towards Mary, increasing the distance d between him and his brother at a constant rate of 10 feet per second ($10\mathrm{ft}/\mathrm{s}$).

a. Set up a Cartesian Coordinate System on a sheet of graph paper. Label and scale each axis. Include units with your labels.
b. Plot the point representing David's initial distance from his brother at time $t = 0$ seconds. Then plot a minimum of 5 additional points using the fact that David's distance from his brother is increasing at a constant rate of 10 feet per second ($10\mathrm{ft}/\mathrm{s}$).
c. Sketch the line representing David's distance from his brother versus time.
d. Find the slope of the line.

In Exercises 5-14, calculate the slope of the line passing through the points P and Q. Be sure to reduce your answer to lowest terms.

5) $P(9,0), Q(-9,15)$

Answer

$$-\frac{5}{6}$$

6) $P(19,-17), Q(-13,19)$

7) $P(0,11), Q(16,-11)$

Answer

$$-\frac{11}{8}$$

8) $P(-10,-8), Q(11,19)$

9) $P(11,1), Q(-1,-1)$

Answer

$$\frac{1}{6}$$

10) $P(16,-15), Q(-11,12)$

11) $P(-18,8), Q(3,-10)$

Answer

$-\frac{6}{7}$

12) $P(9,9), Q(-6,3)$

13) $P(-18,10), Q(-9,7)$

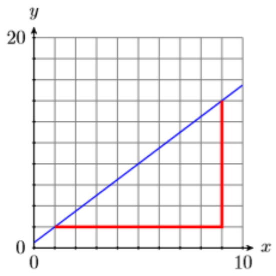
Answer

$-\frac{1}{3}$

14) $P(-7,20), Q(7,8)$

In Exercises 15-18, use the right triangle provided to help determine the slope of the line. Be sure to pay good attention to the scale provided on each axis when counting boxes to determine the change in y and the change in x.

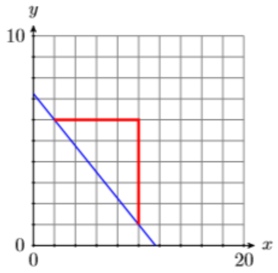
15)



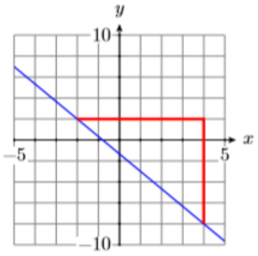
Answer

$\frac{3}{2}$

16)



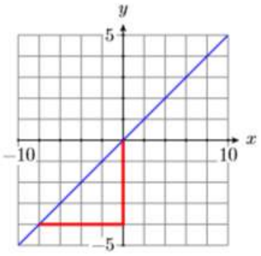
17)



Answer

$-\frac{5}{3}$

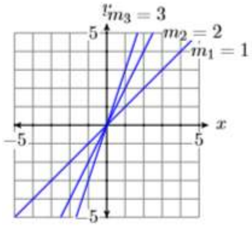
18)



19) On one coordinate system, sketch each of the lines that pass through the following pairs of points. Label each line with its slope, then explain the relationship between the slope found and the steepness of the line drawn.

a. $(0,0)$ and $(1,1)$
b. $(0,0)$ and $(1,2)$
c. $(0,0)$ and $(1,3)$

Answer



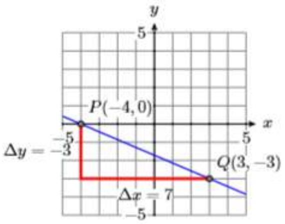
20) On one coordinate system, sketch each of the lines that pass through the following pairs of points. Label each line with its slope, then explain the relationship between the slope found and the steepness of the line drawn.

a. $(0,0)$ and $(1,-1)$
b. $(0,0)$ and $(1,-2)$
c. $(0,0)$ and $(1,-3)$

In Exercises 21-30, setup a coordinate system on graph paper, then sketch the line passing through the point P with the slope m.

21) $P(-4,0), m=-3/7$

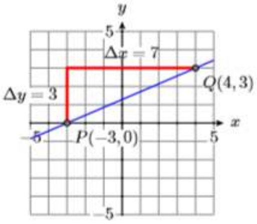
Answer



22) $P(-3,0), m=-3/7$

23) $P(-3,0), m=3/7$

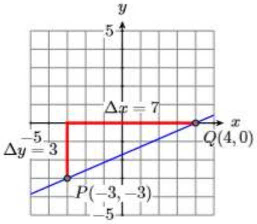
Answer



24) $P(-3,0), m=3/4$

25) $P(-3,-3), m=3/7$

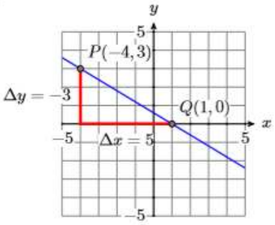
Answer



26) $P(-2,3), m=-3/5$

27) $P(-4,3), m=-3/5$

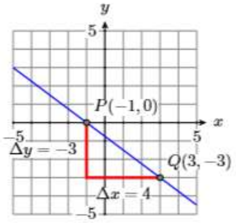
Answer



28) $P(-1,-3), m=3/4$

29) $P(-1,0), m=-3/4$

Answer



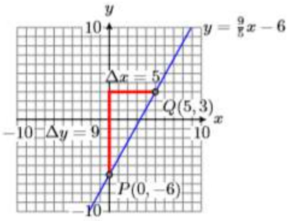
30) $P(-3,3), m=-3/4$

3.4: Slope-Intercept Form of a Line

In Exercises 1-6, setup a coordinate system on graph paper, then sketch the line having the given equation. Label the line with its equation.

1) $y=\frac{9}{5}x-6$

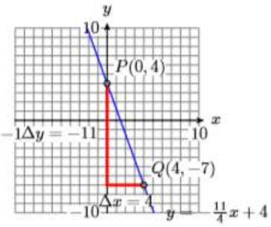
Answer



2) $y=\frac{8}{7}x-1$

3) $y=-\frac{11}{4}x+4$

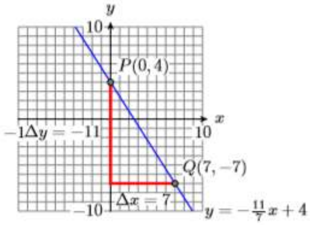
Answer



4) $y=\frac{5}{4}x$

5) $y=-\frac{11}{7}x+4$

Answer

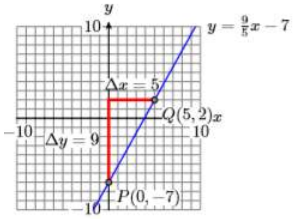


6) $y=-\frac{7}{5}x+7$

In Exercises 7-12, sketch the line with given y-intercept slope. Label the line with the slope-intercept form of its equation.

7) $(0,-7), 9/5$

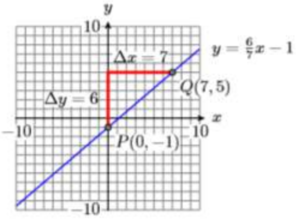
Answer



8) $(0,7), -4/5$

9) $(0,-1), 6/7$

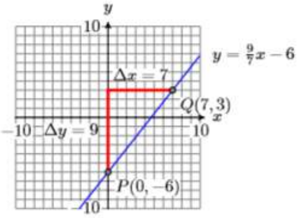
Answer



10) $(0,1), -7/5$

11) $(0,-6), 9/7$

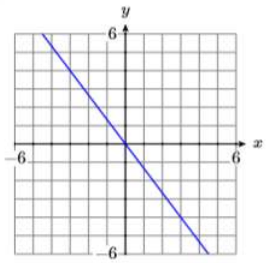
Answer



12) $(0,-5), 7/5$

In Exercises 13-20, determine the equation of the given line in slope-intercept form.

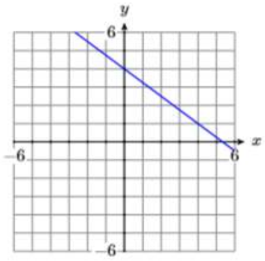
13)



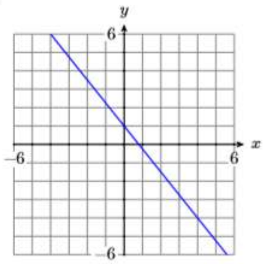
Answer

$$y = -\frac{4}{3}x$$

14)



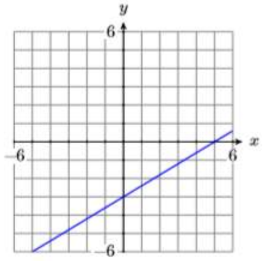
15)



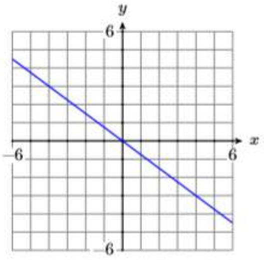
Answer

$y=-\frac{5}{4}x+1$

16)



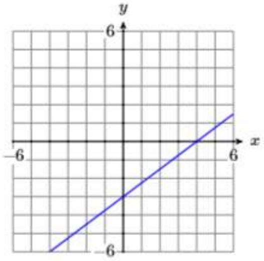
17)



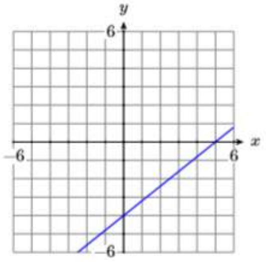
Answer

$y=-\frac{3}{4}x$

18)



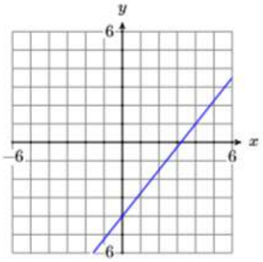
19)



Answer

$$y = \frac{4}{5}x - 4$$

20)



21) Assume that the relationship between an object's velocity and its time is linear. At $t = 0$ seconds, the object's initial velocity is 20m/s. It then begins to speed up at a constant rate of 5 meters per second per second.

a. Set up a coordinate system, placing the time t on the horizontal axis and the velocity v on the vertical axis. Label and scale each axis. Include units in your labels.
b. Use the initial velocity and the rate at which the object's velocity is increasing to draw the line representing the object's velocity at time t. Use the slope-intercept form to determine the equation of the line.
c. Replace x and y in the equation found in part (b) with t and v, respectively. Use the result to determine the velocity of the object after 14 seconds.

Answer

c) 90m/s

22) Assume that the relationship between an object's velocity and its time is linear. At $t = 0$ seconds, the object's initial velocity is 80m/s. It then begins to lose speed at a constant rate of 4 meters per second per second.

a. Set up a coordinate system, placing the time t on the horizontal axis and the velocity v on the vertical axis. Label and scale each axis. Include units in your labels.
b. Use the initial velocity and the rate at which the object's velocity is increasing to draw the line representing the object's velocity at time t. Use the slope-intercept form to determine the equation of the line.
c. Replace x and y in the equation found in part (b) with t and v, respectively. Use the result to determine the velocity of the object after 13 seconds.

23) A water tank initially (at time $t = 0$ min) contains 100 gallons of water. A pipe is opened and water pours into the tank at a constant rate of 25 gallons per minute. Assume that the relation between the volume V of water in the tank and time t is linear.

a. Set up a coordinate system, placing the time t on the horizontal axis and the volume of water V on the vertical axis. Label and scale each axis. Include units in your labels.
b. Use the initial volume of water in the tank and the rate at which the volume of water is increasing to draw the line representing the volume V of water in the tank at time t. Use the slope-intercept form to determine the equation of the line.

c. Replace x and y in the equation found in part (b) with t and V, respectively. Use the result to predict how much time must pass until the volume of water in the tank reaches 400 gallons.

Answer

c) 12 min

24) A water tank initially (at time $t = 0$ min) contains 800 gallons of water. A spigot is opened at the bottom of the tank and water pours out at a constant rate of 40 gallons per minute. Assume that the relation between the volume V of water in the tank and time t is linear.

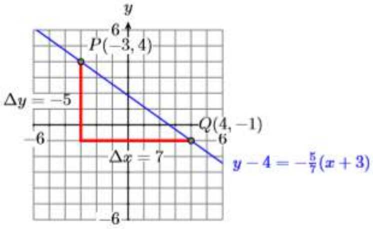
a. Set up a coordinate system, placing the time t on the horizontal axis and the volume of water V on the vertical axis. Label and scale each axis. Include units in your labels.
b. Use the initial volume of water in the tank and the rate at which the volume of water is decreasing to draw the line representing the volume V of water in the tank at time t. Use the slope-intercept form to determine the equation of the line.
c. Replace x and y in the equation found in part (b) with t and V, respectively. Use the result to predict how much time must pass until the water tank is empty.

3.5: Point-Slope Form of a Line

In Exercises 1-6, set up a coordinate system on a sheet of graph paper, then sketch the line through the given point with the given slope. Label the line with its equation in point-slope form.

1) $m = -5/7, P(-3, 4)$

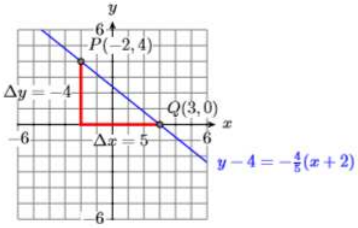
Answer



2) $m = 3/4, P(-2, -4)$

3) $m = -4/5, P(-2, 4)$

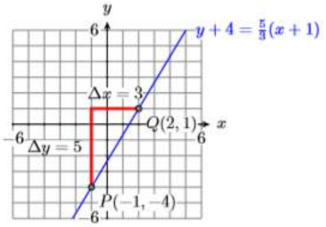
Answer



4) $m = 5/6, P(-3, -1)$

5) $m = 5/3, P(-1, -4)$

Answer

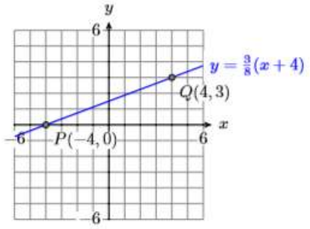


6) $m = -3/8, P(-4, 0)$

In Exercises 7-12, set up a coordinate system on a sheet of graph paper, then sketch the line through the given points. Label the line with the point-slope form of its equation.

7) $P(-4, 0)$ and $Q(4, 3)$

Answer

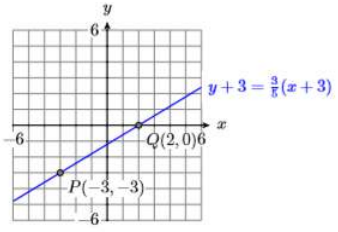


Alternate answer: $y - 3 = \frac{3}{8}(x - 4)$

8) $P(-2, 4)$ and $Q(2, -1)$

9) $P(-3, -3)$ and $Q(2, 0)$

Answer

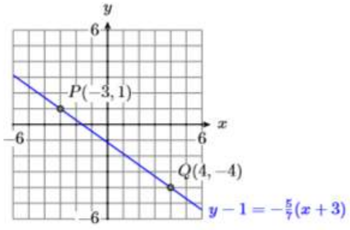


Alternate answer: $y = \frac{3}{5}(x - 2)$

10) $P(-3, 4)$ and $Q(2, 0)$

11) $P(-3, 1)$ and $Q(4, -4)$

Answer



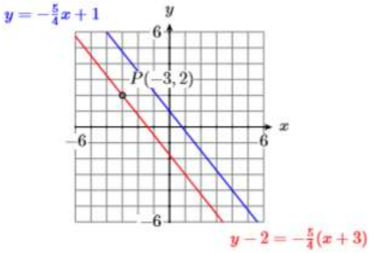
Alternate answer: $y + 4 = -\frac{5}{7}(x - 4)$

12) $P(-1, 0)$ and $Q(4, 3)$

In Exercises 13-18, on a sheet of graph paper, sketch the given line. Plot the point P and draw a line through P that is parallel to the first line. Label this second line with its equation in point-slope form.

13) $y=-\frac{5}{4}x+1, P(-3,2)$

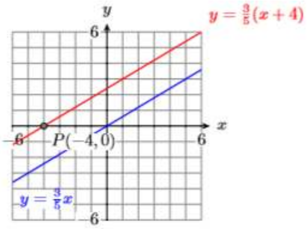
Answer



14) $y=-\frac{3}{5}x, P(-4,0)$

15) $y=\frac{3}{5}x, P(-4,0)$

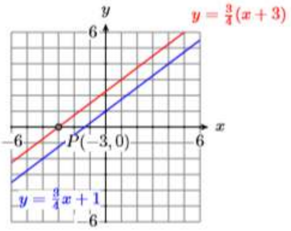
Answer



16) $y=\frac{5}{3}x-1, P(-2,-2)$

17) $y=\frac{3}{4}x+1, P(-3,0)$

Answer

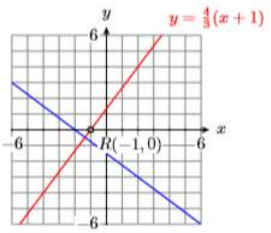


18) $y=\frac{5}{4}x-4, P(-2,-2)$

In Exercises 19-24, on a sheet of graph paper, sketch the line passing through the points P and Q. Plot the point R and draw a line through R that is perpendicular to the line passing through the points P and Q. Label this second line with its equation in point-slope form.

19) $P(-2,0), Q(2,-3)$, and $R(-1,0)$

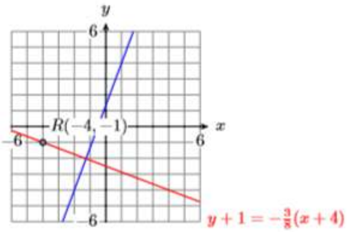
Answer



20) $P(-1,3), Q(2,-2)$, and $R(-1,0)$

21) $P(-2,-4), Q(1,4)$, and $R(-4,-1)$

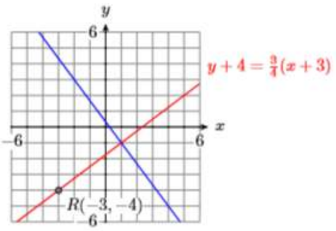
Answer



22) $P(-4,-4), Q(-1,3)$, and $R(-4,2)$

23) $P(-2,3), Q(1,-1)$, and $R(-3,-4)$

Answer



24) $P(-4,4), Q(1,-4)$, and $R(-4,-3)$

25) Assume that the relationship between an object's velocity and its time is linear. At 3 seconds, the object's velocity is $50 \mathrm{ft} / \mathrm{s}$. At 14 seconds, the object's velocity is $30 \mathrm{ft} / \mathrm{s}$.

a. Set up a coordinate system, placing the time t on the horizontal axis and the velocity v on the vertical axis. Label and scale each axis. Include units in your labels.
b. Plot the points determined by the given data and draw a line through them. Use the point-slope form of the line to determine the equation of the line.
c. Replace x and y in the equation found in part (b) with t and v, respectively, then solve the resulting equation for v.
d. Use the result of part (c) to determine the velocity of the object after 6 seconds.

Answer

$44.5 \mathrm{ft} / \mathrm{s}$

26) Water freezes at approximately $32^{\circ} \mathrm{F}$ and $273^{\circ} \mathrm{K}$, where F represents the temperature measured on the Fahrenheit scale and K represents the temperature measured on the Kelvin scale. Water boils at approximately $212^{\circ} \mathrm{F}$ and $373^{\circ} \mathrm{K}$. Assume that the relation between the Fahrenheit and Kelvin temperatures is linear.

a. Set up a coordinate system, placing the Kelvin temperature K on the horizontal axis and the Fahrenheit temperature F on the vertical axis. Label and scale each axis. Include units in your labels.

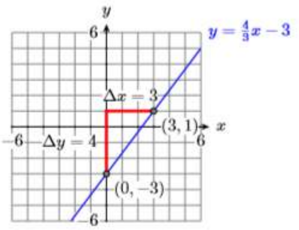
b. Plot the points determined by the given data and draw a line through them. Use the point-slope form of the line to determine the equation of the line.
c. Replace x and y in the equation found in part (b) with K and F, respectively, then solve the resulting equation for F.
d. Use the result of part (c) to determine the Fahrenheit temperature of the object if the Kelvin temperature is $212^{\circ}\mathrm{K}$.

3.6: Standard Form of a Line

In Exercises 1-6, place the given standard form into slope-intercept form and sketch its graph. Label the graph with its slope-intercept form.

1) $4x-3y=9$

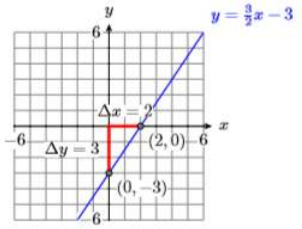
Answer



2) $2x-3y=3$

3) $3x-2y=6$

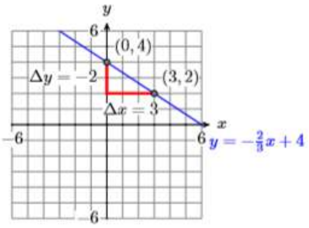
Answer



4) $5x-3y=3$

5) $2x+3y=12$

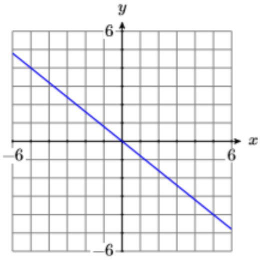
Answer



6) $3x+4y=8$

In Exercises 7-10, determine an equation of the given line in standard form.

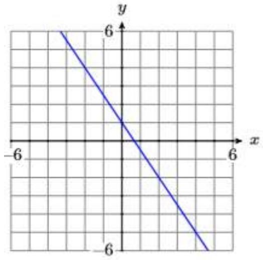
7)



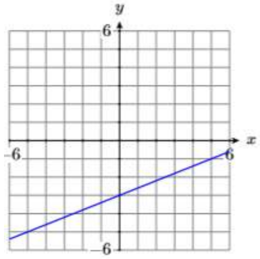
Answer

$4x + 5y = 0$

8)



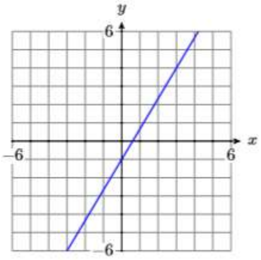
9)



Answer

$2x - 5y = 15$

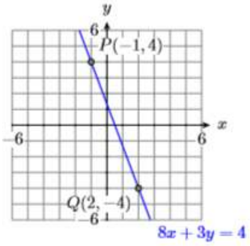
10)



In Exercises 11-16, sketch the line passing through the points P and Q. Label the line with its equation in standard form.

11) $P(-1,4)$ and $Q(2,-4)$

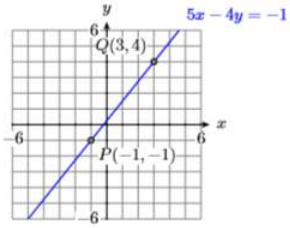
Answer



12) $P(-1,4)$ and $Q(3,1)$

13) $P(-1,-1)$ and $Q(3,4)$

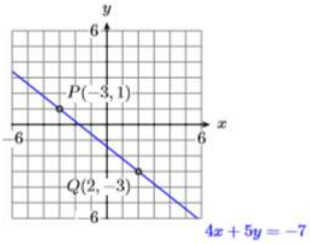
Answer



14) $P(2,-1)$ and $Q(4,4)$

15) $P(-3,1)$ and $Q(2,-3)$

Answer

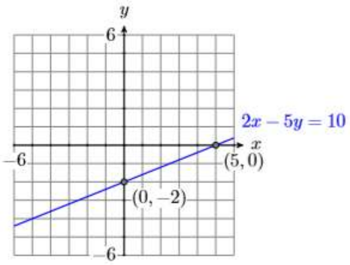


16) $P(-4,3)$ and $Q(0,0)$

In Exercises 17-22, plot the x- and y-intercepts of the line having the given equation, then draw the line through the intercepts and label it with its equation.

17) $2x-5y=10$

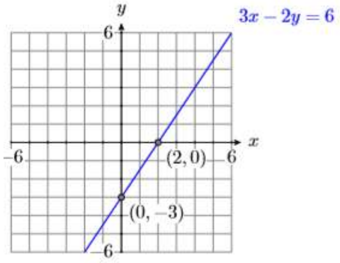
Answer



18) $2x+3y=-6$

19) $3x-2y=6$

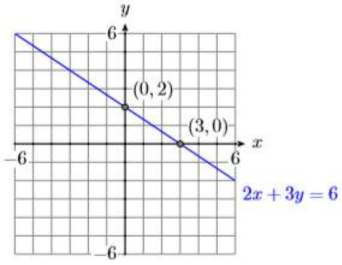
Answer



20) $3x-4y=12$

21) $2x+3y=6$

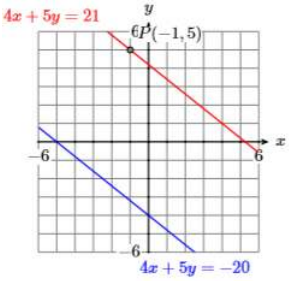
Answer



22) $2x-3y=-6$

23) Find an equation of the line (in standard form) that passes through the point $P(-1,5)$ that is parallel to the line $4x+5y=-20$.

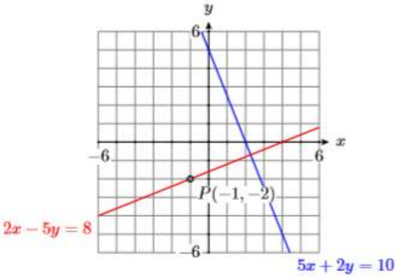
Answer



24) Find an equation of the line (in standard form) that passes through the point $P(-3,2)$ that is parallel to the line $3x+5y=-15$.

25) Find an equation of the line (in standard form) that passes through the point $P(-1,-2)$ that is perpendicular to the line $5x+2y=10$.

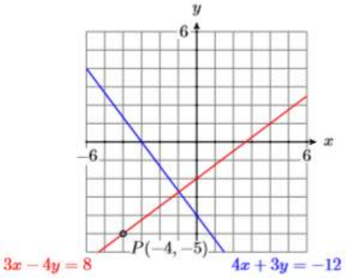
Answer



26) Find an equation of the line (in standard form) that passes through the point $P(-1,-2)$ that is perpendicular to the line $2x+5y=-10$.

27) Find an equation of the line (in standard form) that passes through the point $P(-4,-5)$ that is perpendicular to the line $4x+3y=-12$.

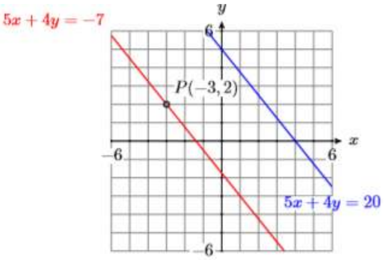
Answer



28) Find an equation of the line (in standard form) that passes through the point $P(-2,-3)$ that is perpendicular to the line $3x+2y=6$.

29) Find an equation of the line (in standard form) that passes through the point $P(-3,2)$ that is parallel to the line $5x+4y=20$.

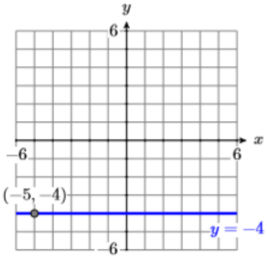
Answer



30) Find an equation of the line (in standard form) that passes through the point $P(-1,3)$ that is parallel to the line $3x+5y=-15$.

31) Sketch the equation of the horizontal line passing through the point $P(-5,-4)$ and label it with its equation.

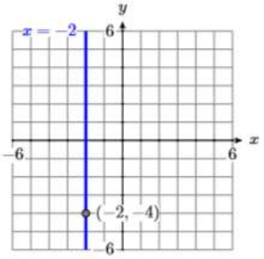
Answer



32) Sketch the equation of the vertical line passing through the point $P(-4,4)$ and label it with its equation.

33) Sketch the equation of the vertical line passing through the point $P(-2,-4)$ and label it with its equation.

Answer



34) Sketch the equation of the vertical line passing through the point $P(1,3)$ and label it with its equation.

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CHAPTER OVERVIEW

4: Systems of Linear Equations

4.1: Solving Systems by Graphing
4.2: Solving Systems by Substitution
4.3: Solving Systems by Elimination
4.4: Applications of Linear Systems
4.E: Systems of Linear Equations (Exercises)

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4.1: Solving Systems by Graphing

In this section we introduce a graphical technique for solving systems of two linear equations in two unknowns. As we saw in the previous chapter, if a point satisfies an equation, then that point lies on the graph of the equation. If we are looking for a point that satisfies two equations, then we are looking for a point that lies on the graphs of both equations; that is, we are looking for a point of intersection.

For example, consider the the two equations:

$$\begin{aligned} x-3y&=-9\\ 2x+3y&=18 \end{aligned}$$

which is called a *system of linear equations*. The equations are linear equations because their graphs are lines, as shown in Figure $4.1.1$. Note that the two lines in Figure $4.1.1$ intersect at the point $(3,4)$. Therefore, the point $(3,4)$ should satisfy both equations. Let’s check.

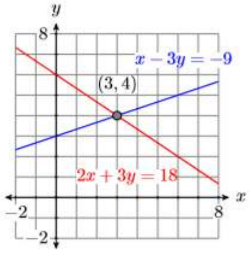


Figure $4.1.1$: The point of intersection is a solution of the system of linear equations.

Substitute 3 for x and 4 for y.

$$\begin{aligned} x-3y&=-9\\ 3-3(4)&=-9\\ 3-12&=-9\\ -9&=-9 \end{aligned}$$

Substitute 3 for x and 4 for y.

$$\begin{aligned} 2x+3y&=18\\ 2(3)+3(4)&=18\\ 6+12&=18\\ 18&=18 \end{aligned}$$

Hence, the point $(3,4)$ satisfies both equations and is called a solution of the system.

Solution of a linear system

A point (x,y) is called a solution of a system of two linear equations if and only if it satisfied both equations. Furthermore, because a point satisfies an equation if and only if it lies on the graph of the equation, to solve a system of linear equations graphically, we need to determine the point of intersection of the two lines having the given equations.

Let’s try an example.

Example $4.1.1$

Solve the following system of equations:

$$\begin{aligned} 3x+2y&=12\\ y&=x+1 \end{aligned} \tag{4.1.1}$$

Solution

We are looking for the point (x,y) that satisfies both equations; that is, we are looking for the point that lies on the graph of both equations. Therefore, the logical approach is to plot the graphs of both lines, then identify the point of intersection.

First, let's determine the x- and y-intercepts of $3x+2y=12$.

To find the x-intercept, let $y=0$.

$$\begin{aligned} 3x+2y&=12 \\ 3x+2(0)&=12 \\ 3x&=12 \\ x&=4 \end{aligned}$$

To find the y-intercept, let $x=0$.

$$\begin{aligned} 3x+2y&=12 \\ 3(0)+2y&=12 \\ 2y&=12 \\ y&=6 \end{aligned}$$

Hence, the x-intercept is $(4,0)$ and the y-intercept is $(0,6)$. These intercepts are plotted in Figure $4.1.2$ and the line $3x+2y=12$ is drawn through them.

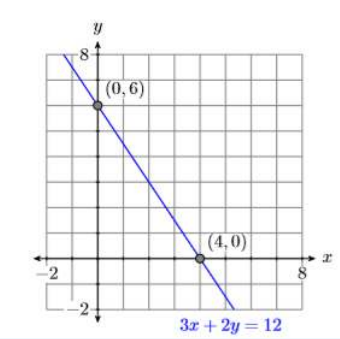


Figure $4.1.2$: Drawing the graph of $3x+2y=12$.

Comparing the second equation $y=x+1$ with the slope-intercept form $y=mx+b$, we see that the slope is $m=1$ and they-intercept is $(0,1)$. Plot the intercept $(0,1)$, then go up 1 unit and right 1 unit, then draw the line (see Figure $4.1.3$).

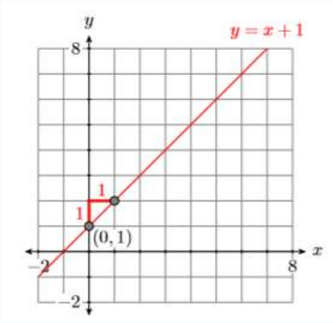


Figure $4.1.3$: Drawing the graph of $y=x+1$.

We are trying to find the point that lies on both lines, so we plot both lines on the same coordinate system, labeling each with its equation (see Figure $4.1.4$). It appears that the lines intersect at the point $(2,3)$, making $(x,y)=(2,3)$ the solution of System in Example $4.1.1$ (see Figure $4.1.4$).

Check: To show that $(x,y)=(2,3)$ is a solution of System $4.1.1$, we must show that we get true statements when we substitute 2 for x and 3 for y in both equations of System $4.1.1$.

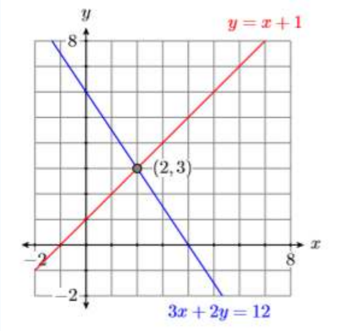


Figure 4.1.4: The coordinates of the point of intersection is the solution of System 4.1.1.

Substituting 2 for x and 3 for y in $3x + 2y = 12$, we get:

$$\begin{aligned} 3x + 2y &= 12 \\ 3(2) + 2(3) &= 12 \\ 6 + 6 &= 12 \\ 12 &= 12 \end{aligned}$$

Hence, $(2, 3)$ satisfies the equation $3x + 2y = 12$.

Substituting 2 for x and 3 for y in $y = x + 1$, we get:

$$\begin{aligned} y &= x + 1 \\ 3 &= 2 + 1 \\ 3 &= 3 \end{aligned}$$

Hence, $(2, 3)$ satisfies the equation $y = x + 1$.

Because $(2, 3)$ satisfies both equations, this makes $(2, 3)$ a solution of System 4.1.1.

Exercise 4.1.1

Solve the following system of equations:

$$\begin{aligned} 2x - 5y &= -10 \\ y &= x - 1 \end{aligned}$$

Answer

$(5, 4)$

Example 4.1.2

Solve the following system of equations:

$$\begin{aligned} 3x - 5y &= -15 \\ 2x + y &= -4 \end{aligned} \tag{4.1.2}$$

Solution

Once again, we are looking for the point that satisfies both equations of the System 4.1.2. Thus, we need to find the point that lies on the graphs of both lines represented by the equations of System 4.1.2. The approach will be to graph both lines, then approximate the coordinates of the point of intersection. First, let's determine the x- and y-intercepts of $3x - 5y = -15$.

To find the x-intercept, let $y = 0$.

$$\begin{aligned} 3x - 5y &= -15 \\ 3x - 5(0) &= -15 \\ 3x &= -15 \\ x &= -5 \end{aligned}$$

To find the y-intercept, let $x = 0$.

$$\begin{aligned} 3x-5y &= -15 \\ 3(0)-5y &= -15 \\ -5y &= -15 \\ y &= 3 \end{aligned}$$

Hence, the x-intercept is $(-5, 0)$ and the y-intercept is $(0, 3)$. These intercepts are plotted in Figure $4.1.5$ and the line $3x-5y = -15$ is drawn through them.

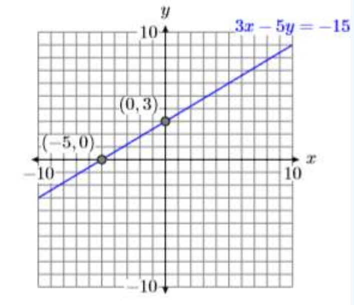


Figure $4.1.5$: Drawing the graph of the line $3x-5y=-15$.

Next, let's determine the intercepts of the second equation $2x+y=-4$.

To find the x-intercept, let $y=0$.

$$\begin{aligned} 2x+y &= -4 \\ 2x+0 &= -4 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

To find the y-intercept, let $x=0$.

$$\begin{aligned} 2x+y &= -4 \\ 2(0)+y &= -4 \\ y &= -4 \end{aligned}$$

Hence, the x-intercept is $(-2, 0)$ and the y-intercept is $(0, -4)$. These intercepts are plotted in Figure $4.1.6$ and the line $2x+y = -4$ is drawn through them.

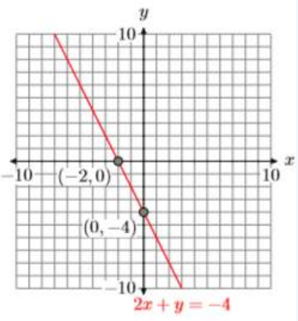


Figure $4.1.6$: Drawing the graph of the line $2x+y=-4$.

To find the solution of System 4.1.2, we need to plot both lines on the same coordinate system and determine the coordinates of the point of intersection. Unlike Example $4.1.1$, in this case we'll have to be content with an approximation of these coordinates. It appears that the coordinates of the point of intersection are approximately $(-2.6, 1.4)$ (see Figure $4.1.7$).

Check: Because we only have an approximation of the solution of the system, we cannot expect the solution to check exactly in each equation. However, we do hope that the solution checks approximately.

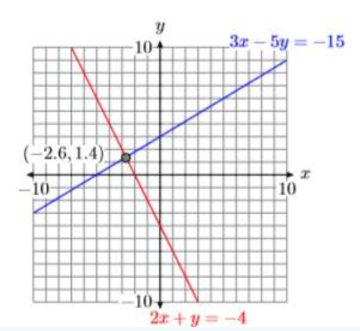


Figure 4.1.7: The approximate coordinates of the point of intersection are $(-2.6, 1.4)$.

Substitute $(x, y)=(-2.6, 1.4)$ into the first equation of System 4.1.2.

$$\begin{aligned} 3x-5y&=-15 \\ 3(-2.6)-5(1.4)&=-15 \\ -7.8-7&=-15 \\ -14.8&=-15 \end{aligned}$$

Note that $(x, y)=(-2.6, 1.4)$ does not check exactly, but it is pretty close to being a true statement.

Substitute $(x, y)=(-2.6, 1.4)$ into the second equation of System 4.1.2.

$$\begin{aligned} 2x+y&=-4 \\ 2(-2.6)+1.4&=-4 \\ -5.2+1.4&=-4 \\ -3.8&=-4 \end{aligned}$$

Again, note that $(x, y)=(-2.6, 1.4)$ does not check exactly, but it is pretty close to being a true statement.

Note

Later in this section we will learn how to use the intersect utility on the graphing calculator to obtain a much more accurate approximation of the actual solution. Then, in Section 4.2 and Section 4.3, we'll show how to find the exact solution.

Exercise 4.1.2

Solve the following system of equations:

$$\begin{aligned} -4x-3y&=12 \\ x-2y&=-2 \end{aligned}$$

Answer

$(-2.7, -0.4)$

Exceptional Cases

Most of the time, given the graphs of two lines, they will intersect in exactly one point. But there are two exceptions to this general scenario.

Example 4.1.3

Solve the following system of equations:

$$\begin{aligned} 2x+3y&=6 \\ 2x+3y&=-6 \end{aligned} \tag{4.1.3}$$

Solution

Let's place each equation in slope-intercept form by solving each equation for y.

Solve $2x+3y=6$ for y:

$$\begin{aligned} 2x+3y&=6 \\ 2x+3y-2x&=6-2x \\ 3y&=6-2x \\ \frac{3y}{3}&=\frac{6-2x}{3} \\ y&=-\frac{2}{3}x+2 \end{aligned}$$

Solve $2x+3y=-6$ for y:

$$\begin{aligned} 2x+3y&=-6 \\ 2x+3y-2x&=-6-2x \\ 3y&=-6-2x \\ \frac{3y}{3}&=\frac{-6-2x}{3} \\ y&=-\frac{2}{3}x-2 \end{aligned}$$

Comparing $y=(-2/3)x+2$ with the slope-intercept form $y=mx+b$ tells us that the slope is $m=-2/3$ and they-intercept is $(0,2)$. Plot the intercept $(0,2)$, then go down 2 units and right 3 units and draw the line (see Figure $4.1.8$).

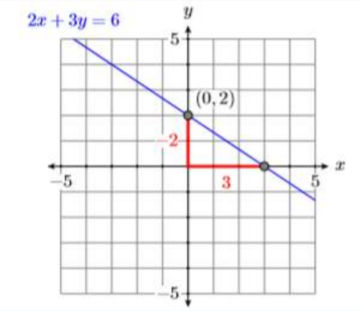


Figure $4.1.8$: Drawing the graph of the line $2x+3y=6$.

Comparing $y=(-2/3)x-2$ with the slope-intercept form $y=mx+b$ tells us that the slope is $m=-2/3$ and they-intercept is $(0,-2)$. Plot the intercept $(0,-2)$, then go down 2 units and right 3 units and draw the line (see Figure $4.1.9$).

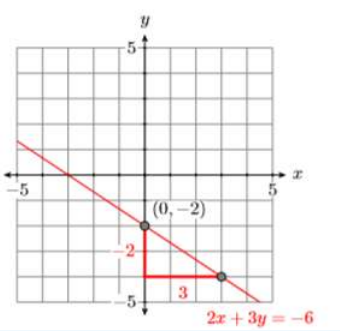


Figure $4.1.9$: Drawing the graph of the line $2x+3y=-6$.

To find the solution of System 4.1.3, draw both lines on the same coordinate system (see Figure $4.1.10$). Note how the lines appear to be parallel (they don't intersect). The fact that both lines have the same slope $-2/3$ confirms our suspicion that the lines are parallel. However, note that the lines have different y-intercepts. Hence, we are looking at two parallel but distinct lines (see Figure $4.1.10$) that do not intersect. Hence, System \ref{system3} has no solution.

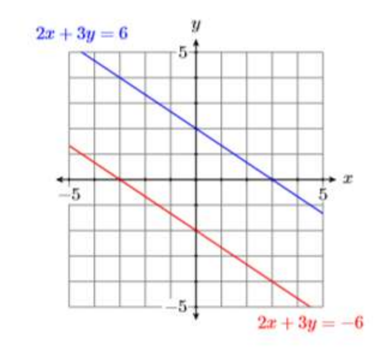


Figure $4.1.10$: The lines $2x+3y=6$ and $2x+3y=-6$ are parallel, distinct lines.

Exercise $4.1.3$

Solve the following system of equations:

$$\begin{aligned} x-y&=3\\ -2x+2y&=4 \end{aligned}$$

Answer

No solution.

Example $4.1.4$

Solve the following system of equations:

$$\begin{aligned} x-y&=3\\ -2x+2y&=-6 \end{aligned} \tag{4.1.4}$$

Solution

Let's solve both equations for y.

Solve $x-y=3$ for y:

$$\begin{aligned} x-y&=3\\ x-y-x&=3-x\\ -y&=-x+3\\ -1(-y)&=-1(-x+3)\\ y&=x-3 \end{aligned}$$

Solve $-2x+2y=-6$ for y:

$$\begin{aligned} -2x+2y&=-6\\ -2x+2y+2x&=-6+2x\\ 2y&=2x-6\\ \frac{2y}{2}&=\frac{2x-6}{2}\\ y&=x-3 \end{aligned}$$

Both lines have slope $m=1$, and both have the same y-intercept $(0,-3)$. Hence, the two lines are identical (see Figure $4.1.11$). Hence, System 4.1.4 has an infinite number of points of intersection. Any point on either line is a solution of the system. Examples of points of intersection (solutions satisfying both equations) are $(0,-3)$, $(1,-2)$, and $(3,0)$.

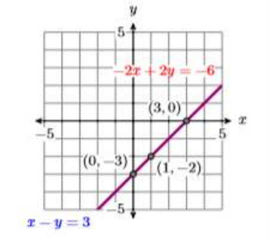


Figure 4.1.11: $x-y=3$ and $-2x+2y=-6$ are the same line.

Alternate solution:

A much easier approach is to note that if we divide both sides of the second equation $-2x+2y=-6$ by -2, we get:

$$\begin{aligned} -2x+2y &= -6 && \text{Second equation in System 4.1.4.} \\ \frac{-2x+2y}{-2} &= \frac{-6}{-2} && \text{Divide both sides by } -2 \\ \frac{-2x}{-2}+\frac{2y}{-2} &= \frac{-6}{-2} && \text{Distribute } -2 \\ x-y &= 3 && \text{Simplify.} \end{aligned}$$

Hence, the second equation in System 4.1.4 is identical to the first. Thus, there are an infinite number of solutions. Any point on either line is a solution.

Exercise 4.1.4

Solve the following system of equations:

$$\begin{aligned} -6x+3y &= -12 \\ 2x-y &= 4 \end{aligned}$$

Answer

There are an infinite number of solutions. The lines are identical, so any point on either line is a solution.

Examples 4.1.1, 4.1.2, 4.1.3, and 4.1.4 lead us to the following conclusion.

Number of solutions of a linear system

When dealing with a system of two linear equations in two unknowns, there are only three possibilities:

1. There is exactly one solution.
2. There are no solutions.
3. There are an infinite number of solutions.

Solving Systems with the Graphing Calculator

We’ve already had experience graphing equations with the graphing calculator. We’ve also used the TRACE button to estimate points of intersection. However, the graphing calculator has a much more sophisticated tool for finding points of intersection. In the next example we’ll use the graphing calculator to find the solution of System 4.1.1 of Example 4.1.1.

Example 4.1.5

Use the graphing calculator to solve the following system of equations:

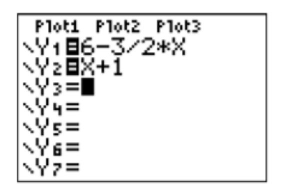
$$\begin{aligned} 3x+2y &= 12 \\ y &= x+1 \end{aligned} \tag{4.1.5}$$

Solution

To enter an equation in the **Y= menu**, the equation must first be solved for y. Hence, we must first solve $3x+2y=12$ for y.

$$\begin{aligned} 3x+2y&=12 && \text{Original equation.}\\ 2y&=12-3x && \text{Subtract } 3x \text{ from both sides of the equation.}\\ \frac{2y}{2}&=\frac{12-3x}{2} && \text{Divide both sides by 2}\\ y&=\frac{12}{2}-\frac{3x}{2} && \text{On the left, simplify. On the right,}\\ y&=6-\frac{3}{2}x && \text{Simplify.} \end{aligned} \tag{4.1.6}$$

We can now substitute both equations of System 4.1.5 into the **Y= menu** (see Figure 4.1.12).



```
Plot1 Plot2 Plot3
\Y1■6-3/2*X
\Y2■X+1
\Y3=■
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 4.1.12: Enter System 4.1.5 equations into the **Y= menu**.

Select **6:ZStandard** from the ZOOM menu to produce the graphs shown in Figure 4.1.13.

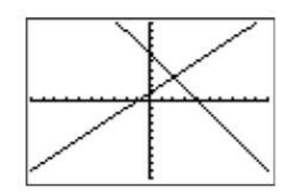


Figure 4.1.13: Select **6:ZStandard** to produce the graphs of the System 4.1.5 equations.

The question now becomes “How do we calculate the coordinates of the point of intersection?” Look on your calculator case just above the TRACE button on the top row of buttons, where you’ll see the word CAlC, painted in the same color as the **2ND** key. Press the **2ND** key, then the TRACE button, which will open the **CALCULATE** menu shown in Figure 4.1.14.

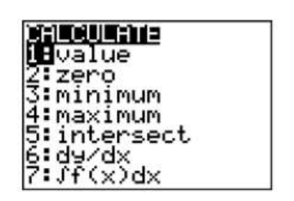


Figure 4.1.14: Press **2ND**, then **TRACE** to open the **CALCULATE** menu. Then select **5:intersect** to produce the screen in Figure 4.1.15.

Note

Having the calculator ask “First curve,” “Second curve,” when there are only two curves on the screen may seem annoying. However, imagine the situation when there are three or more curves on the screen. Then these questions make good sense. You can change your selection of “First curve” or “Second curve” by using the up-and-down arrow keys to move the cursor to a different curve.

Select **5:intersect**. The result is shown in Figure 4.1.15. The calculator has placed the cursor on the curve $y=6-(3/2)x$ (see upper left corner of your viewing screen), and in the lower left corner the calculator is asking you if you want to use the selected curve as the “First curve.” Answer “yes” by pressing the **ENTER** button.

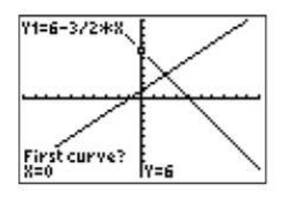

Y1=6-3/2*X
First curve?
X=0 Y=6

Figure 4.1.15: Press the ENTER key on your calculator to say “yes” to the “First curve” selection.

The calculator responds as shown Figure 4.1.16. The cursor jumps to the curve $y = x + 1$ (see upper left corner of your viewing window), and in the lower left corner the calculator is asking you if you want to use the selected curve as the “Second curve.” Answer “yes” by pressing the **ENTER** key again.

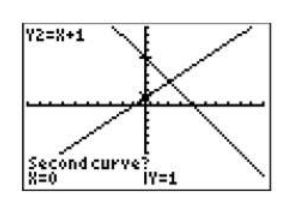

Y2=X+1
Second curve?
X=0 Y=1

Figure 4.1.16: Press the **ENTER** key on your calculator to say “yes” to the “Second curve” selection.

The calculator responds as shown Figure 4.1.17, asking you to “Guess.” In this case, leave the cursor where it is and press the **ENTER** key again to signal the calculator that you are making a guess at the current position of the cursor.

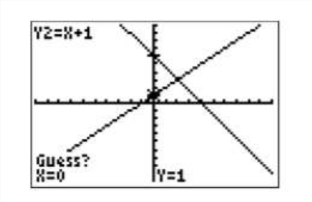

Y2=X+1
Guess?
X=0 Y=1

Figure 4.1.17: Press the **ENTER** key to signal the calculator that you are satisfied with the current position of the cursor as your guess.

The result of pressing **ENTER** to the “Guess” question in Figure 4.1.17 is shown in Figure 4.1.18, where the calculator now provides an approximation of the the coordinates of the intersection point on the bottom edge of the viewing window. Note that the calculator has placed the cursor on the point of intersection in Figure 4.1.17 and reports that the approximate coordinates of the point of intersection are $(2, 3)$.

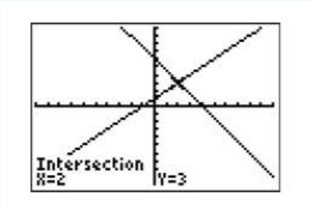

Intersection
X=2 Y=3

Figure 4.1.18: Read the approximate coordinates of the point of intersection along the bottom edge of the viewing window.

Note

In later sections, when we investigate the intersection of two graphs having more than one point of intersection, guessing will become more important. In those future cases, we’ll need to use the left-and-right arrow keys to move the cursor near the point of intersection we wish the calculator to find.

Reporting your solution on your homework. In reporting your solution on your homework paper, follow the Calculator Submission Guidelines from Chapter 3, Section 2. Make an accurate copy of the image shown in your viewing window. Label your axes x and y. At the end of each axis, put the appropriate value of $\mathrm{Xmin}, \mathrm{Xmax}, \mathrm{Ymin}$, and Ymax reported in your calculator’s **WINDOW** menu. Use a ruler to draw the lines and label each with their equations. Finally, label the point of intersection with its coordinates (see Figure 4.1.19). Unless instructed otherwise, always report every single digit displayed on your calculator.

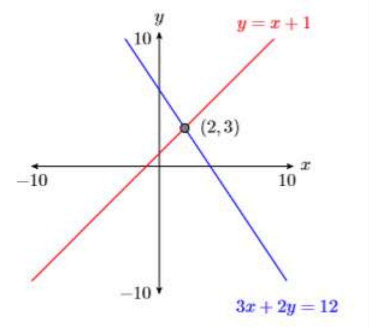


Figure 4.1.19: Reporting your result on your homework paper.

Exercise 4.1.5

Solve the following system of equations:

$$\begin{aligned} 2x - 5y &= 9 \\ y &= 2x - 5 \end{aligned}$$

Answer

$(2, -1)$

Sometimes you will need to adjust the parameters in the **WINDOW** menu so that the point of intersection is visible in the viewing window.

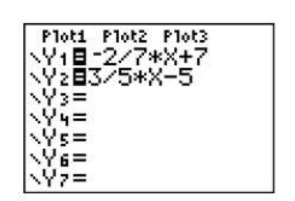
Example 4.1.6

Use the graphing calculator to find an approximate solution of the following system:

$$\begin{aligned} y &= -\frac{2}{7}x + 7 \\ y &= \frac{3}{5}x - 5 \end{aligned} \tag{4.1.7}$$

Solution

Each equation of System 4.1.7 is already solved for y, so we can proceed directly and enter them in the **Y= menu**, as shown in Figure 4.1.20. Select **6:ZStandard** from the **ZOOM** menu to produce the image shown in Figure 4.1.21.



```
Plot1 Plot2 Plot3
\Y1■-2/7*X+7
\Y2■3/5*X-5
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 4.1.20: Enter the equations of System 4.1.7.

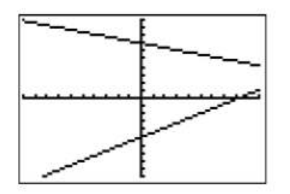
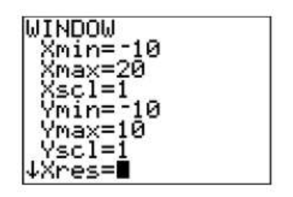


Figure 4.1.21: Select **6:ZStandard** to produce this window.

Obviously, the point of intersection is off the screen to the right, so we'll have to increase the value of Xmax (set Xmax = 20) as shown in Figure 4.1.22. Once you have made that change to Xmax, press the **GRAPH** button to produce the image shown in

Figure 4.1.23.



```
WINDOW
 Xmin=-10
 Xmax=20
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
↓Xres=■
```

Figure 4.1.22: Change Xmax to 20.

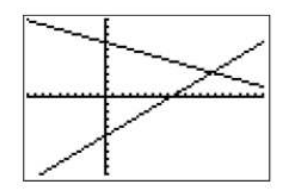


Figure 4.1.23: Press the **GRAPH** button to produce this window.

Now that the point of intersection is visible in the viewing window, press **2ND CALC** and select **5:intersect** from the CALCULATE menu (see Figure 4.1.24). Make three consecutive presses of the **ENTER** button to respond to "First curve," "Second curve," and "Guess." The calculator responds with the image in Figure 4.1.25. Thus, the solution of System 4.1.7 is approximately $(x, y) \approx (13.54837, 3.1290323)$.

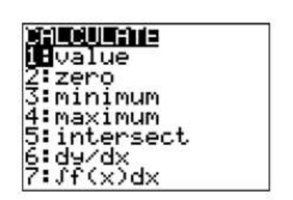
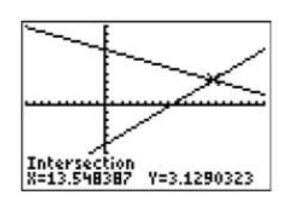


Figure 4.1.24: Press **2ND CALC** to open the **CALCULATE** menu. Select **5:intersect** to find the point of intersection.



```
Intersection
X=13.548387  Y=3.1290323
```

Figure 4.1.25: Three consecutive presses of the **ENTER** key produce the coordinates shown at the bottom of the viewing window.

Warning!

Your calculator is an approximating machine. It is quite likely that your solutions might differ slightly from the solution presented in Figure 4.1.25 in the last $2-3$ places.

Reporting your solution on your homework:

In reporting your solution on your homework paper, follow the Calculator Submission Guidelines from Chapter 3, Section 2. Make an accurate copy of the image shown in your viewing window. Label your axes x and y. At the end of each axis, put the appropriate value of Xmin, Xmax, Ymin, and Ymax reported in your calculator's **WINDOW** menu. Use a ruler to draw the lines and label each with their equations. Finally, label the point of intersection with its coordinates (see Figure 4.1.26). Unless instructed otherwise, always report every single digit displayed on your calculator.

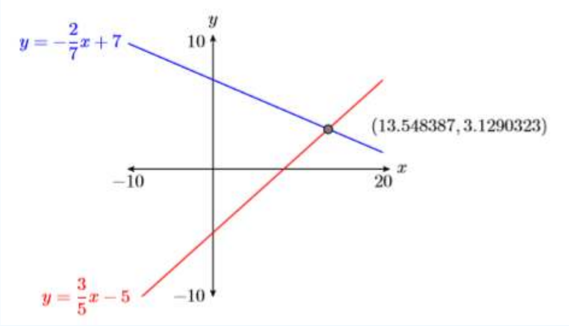


Figure 4.1.26: Reporting your result on your homework paper.

Exercise 4.1.6

Solve the following system of equations:

$$y = \frac{3}{2}x + 6$$
$$y = -\frac{6}{7}x - 4$$

Answer

$(-4.2, -0.4)$

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4.2: Solving Systems by Substitution

In this section we introduce an algebraic technique for solving systems of two equations in two unknowns called the substitution method. The substitution method is fairly straightforward to use. First, you solve either equation for either variable, then substitute the result into the other equation. The result is an equation in a single variable. Solve that equation, then substitute the result into any of the other equations to find the remaining unknown variable.

Example 4.2.1

Solve the following system of equations:

$$2x-5y=-8 \tag{4.2.1}$$

$$y=3x-1 \tag{4.2.2}$$

Solution

Equation 4.2.2 is already solved for y. Substitute Equation 4.2.2 into Equation 4.2.1. This means we will substitute $3x-1$ for y in Equation 4.2.1.

$$\begin{aligned} 2x-5y&=-8 && \text{Equation 4.2.1} \\ 2x-5(3x-1)&=-8 && \text{Substitute } 3x-1 \text{ for } y \text{ in 4.2.1} \end{aligned}$$

Now solve for x.

$$\begin{aligned} 2x-15x+5&=-8 && \text{Distribute } -5 \\ -13x+5&=-8 && \text{Simplify.} \\ -13x&=-13 && \text{Subtract 5 from both sides,} \\ x&=1 && \text{Divide both sides by } -13 \end{aligned}$$

As we saw in Solving Systems by Graphing, the solution to the system is the point of intersection of the two lines represented by the equations in the system. This means that we can substitute the answer $x=1$ into either equation to find the corresponding value of y. We choose to substitute 1 for x in Equation 4.2.2, then solve for y, but you will get exactly the same result if you substitute 1 for x in Equation 4.2.1.

$$\begin{aligned} y&=3x-1 && \text{Equation 4.2.2} \\ y&=3(1)-1 && \text{Substitute 1 for } x \\ y&=2 && \text{Simplify.} \end{aligned}$$

Hence, $(x,y)=(1,2)$ is the solution of the system.

Check: To show that the solution $(x,y)=(1,2)$ is a solution of the system, we need to show that $(x,y)=(1,2)$ satisfies both equations 4.2.1 and 4.2.2.

Substitute $(x,y)=(1,2)$ in Equation 4.2.1:

$$\begin{aligned} 2x-5y&=-8 \\ 2(1)-5(2)&=-8 \\ 2-10&=-8 \\ -8&=-8 \end{aligned}$$

Thus, (1,2) satisfies Equation 4.2.1.

Substitute $(x,y)=(1,2)$ in Equation 4.2.2:

$$\begin{aligned} y&=3x-1 \\ 2&=3(1)-1 \\ 2&=3-1 \\ 2&=2 \end{aligned}$$

Thus, (1,2) satisfies Equation 4.2.2.

Because $(x,y)=(1,2)$ satisfies both equations, it is a solution of the system.

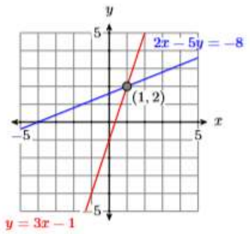


Figure 4.27: $2x - 5y = -8$ and $y = x - 3$ intersect at $(1, 2)$.

Figure 4.2.1: $2x - 5y = -8$ and $y = x - 3$ intersect at $(1, 2)$.

Exercise 4.2.1

Solve the following system of equations:

$$\begin{aligned} 9x + 2y &= -19 \\ y &= 13 + 3x \end{aligned}$$

Answer

$(-3, 4)$

Substitution Method

The substitution method involves these steps:

1. Solve either equation for either variable.
2. Substitute the result from step one into the other equation. Solve the resulting equation.
3. Substitute the result from step two into either of the original system equations or the resulting equation from step one (whichever seems easiest), then solve to find the remaining unknown variable.

Example 4.2.2

Solve the following system of equations:

$$5x - 2y = 12 \tag{4.2.3}$$

$$4x + y = 6 \tag{4.2.4}$$

Solution

The first step is to solve either equation for either variable. This means that we can solve the first equation for x or y, but it also means that we could first solve the second equation for x or y. Of these four possible choices, solving the second Equation 4.2.4 for y seems the easiest way to start.

$$\begin{aligned} 4x + y &= 6 && \text{Equation 4.2.4} \\ y &= 6 - 4x && \text{Subtract } 4x \text{ from both sides.} \end{aligned}$$

Next, substitute $6 - 4x$ for y in Equation 4.2.3.

$$\begin{aligned} 5x - 2y &= 12 && \text{Equation 4.2.3} \\ 5x - 2(6 - 4x) &= 12 && \text{Substitute } 6 - 4x \text{ for } y \text{ in 4.2.3} \\ 5x - 12 + 8x &= 12 && \text{Distribute } -2 \\ 13x - 12 &= 12 && \text{Simplify.} \\ 13x &= 24 && \text{Add 12 to both sides.} \\ x &= \frac{24}{13} && \text{Divide both sides by 13} \end{aligned}$$

Finally, to find the y-value, substitute $24/13$ for x in the equation $y = 6 - 4x$ (you can also substitute $24/13$ for x in equations 4.2.3 or 4.2.4).

$$\begin{aligned} y &= 6 - 4x \\ y &= 6 - 4\left(\frac{24}{13}\right) && \text{Substitute } 24/13 \text{ for } x \text{ in } y = 6 - 4x \\ y &= \frac{78}{13} - \frac{96}{13} && \text{Multiply, then make equivalent fractions.} \\ y &= -\frac{18}{13} && \text{Simplify.} \end{aligned}$$

Hence, $(x, y) = (24/13, -18/13)$ is the solution of the system.

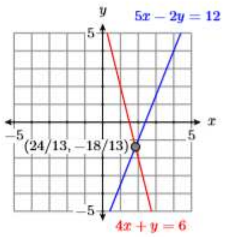


Figure 4.28: $5x - 2y = 12$ and $4x + y = 6$ intersect at $(24/13, -18/13)$.

Figure 4.2.2: $5x - 2y = 12$ and $4x + y = 6$ intersect at $(24/13, -18/13)$.

Check: Let's use the graphing calculator to check the solution. First, we store $24/13$ in X with the following keystrokes (see the result in Figure 4.2.3).

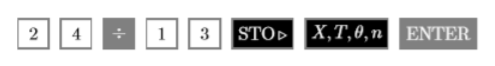
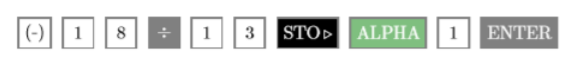
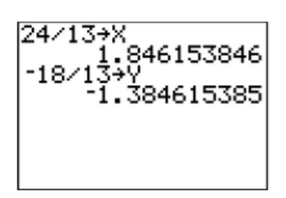


Figure 4.2.3).





```
24/13→X
          1.846153846
-18/13→Y
         -1.384615385
```

Figure 4.2.3: Storing $24/13$ and $-18/13$ in X and Y.

Now, clear the calculator screen by pressing the **CLEAR** button, then enter the left-hand side of Equation 4.2.3 with the following keystrokes (see the result in Figure 4.2.4).

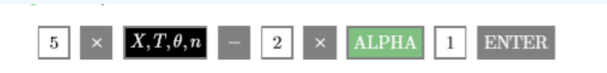
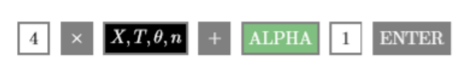
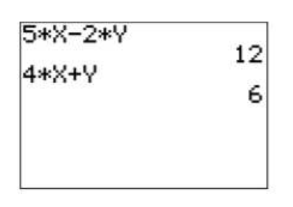


Figure 4.2.4). Note that each left-hand side produces the number on the right-hand sides of equations 4.2.3 and 4.2.4. Thus, the solution $(x, y) = (24/13, -18/13)$ checks.





```
5*X-2*Y
                12
4*X+Y
                 6
```

Figure 4.2.4: Checking equations 4.2.3 and 4.2.4.

Exercise 4.2.2

Solve the following system of equations:

$$\begin{aligned} x-2y&=13\\ 4x-3y&=26 \end{aligned}$$

Answer

$(13/5,-26/5)$

Example 4.2.3

Solve the following system of equations:

$$3x-2y=6 \tag{4.2.5}$$

$$4x+5y=20 \tag{4.2.6}$$

Solution

Dividing by -2 gives easier fractions to deal with than dividing by 3, 4, or 5, so let's start by solving equation (4.2.5) for y.

$$\begin{aligned} 3x-2y&=6 && \text{Equation 4.2.5}\\ -2y&=6-3x && \text{Subtract } 3x \text{ from both sides.}\\ y&=\frac{6-3x}{-2} && \text{Divide both sides by } -2\\ y&=-3+\frac{3}{2}x && \text{Divide both 6 and } -3x \text{ by } -2 \text{ using distributive property.} \end{aligned}$$

Substitute $-3+\frac{3}{2}x$ for y in Equation 4.2.6

$$\begin{aligned} 4x+5y&=20 && \text{Equation 4.2.6}\\ 4x+5\left(-3+\frac{3}{2}x\right)&=20 && \text{Substitute } -3+\frac{3}{2}x \text{ for } y\\ 4x-15+\frac{15}{2}x&=20 && \text{Distribute the 5}\\ 8x-30+15x&=40 && \text{Clear fractions by multiplying}\\ 23x&=70 && \text{Simplify. Add 30 to both sides.}\\ x&=\frac{70}{23} && \text{Divide both sides by 23} \end{aligned}$$

To find y, substitute $70/23$ for x into equation $y=-3+\frac{3}{2}x$. You could also substitute $70/23$ for x in equations 4.2.5 or 4.2.6 and get the same result.

$$\begin{aligned} y&=-3+\frac{3}{2}x\\ y&=-3+\frac{3}{2}\left(\frac{70}{23}\right) && \text{Substitute } 70/23 \text{ for } x\\ y&=-\frac{69}{23}+\frac{105}{23} && \text{Multiply. Make equivalent fractions.}\\ y&=\frac{36}{23} && \text{Simplify.} \end{aligned}$$

Hence, $(x,y)=(70/23,36/23)$ is the solution of the system.

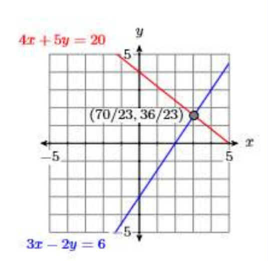
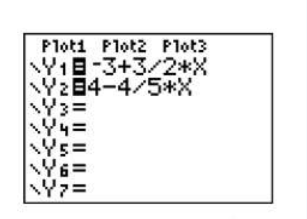


Figure 4.2.5: $3x - 2y = 6$ and $4x + 5y = 20$ intersect at $(70/23, 36/23)$.

Check: To check this solution, let's use the graphing calculator to find the solution of the system. We already know that $3x - 2y = 6$ is equivalent to $y = -3 + \frac{3}{2}x$. Let's also solve Equation 4.2.6 for y.

$$\begin{aligned} 4x + 5y &= 20 && \text{Equation 4.2.6} \\ 5y &= 20 - 4x && \text{Subtract } 4x \text{ from both sides.} \\ y &= \frac{20 - 4x}{5} && \text{Divide both sides by 5} \\ y &= 4 - \frac{4}{5}x && \text{Divide both 20 and } -4x \text{ by 5 using the distributive property.} \end{aligned}$$

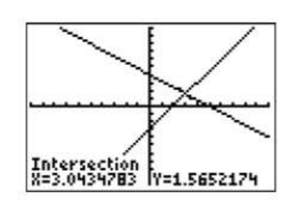
Enter $y = -3 + \frac{3}{2}x$ and $y = 4 - \frac{4}{5}x$ into the **Y**= menu of the graphing calculator (see Figure 4.32).



```
Plot1 Plot2 Plot3
\Y1■-3+3/2*X
\Y2■4-4/5*X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 4.2.6: Enter$y - 3 + \frac{3}{2}x$ and $y = 4 - \frac{4}{5}x$ in **Y1** and **Y2**, respectively.

Press the **ZOOM** button and select **6:ZStandard**. Press **2ND CALC** to open the **CALCULATE** menu, select **5:intersect**, then press the **ENTER** key three times in succession to enter "Yes" to the queries "First curve," "Second curve," and "Guess." The result is shown in Figure 4.2.7.



```
Intersection
X=3.0434783    Y=1.5652174
```

Figure 4.2.7: Use **5:intersect** on the **CALC** menu to calculate the point of intersection.

At the bottom of the viewing window in Figure 4.2.7, note how the coordinates of the point of intersection are stored in the variables **X** and **Y**. Without moving the cursor, (the variables **X** and **Y** contain the coordinates of the cursor), quit the viewing window by pressing **2ND QUIT**, which is located above the **MODE** key. Then press the **CLEAR** button to clear the calculator screen.

Now press the $\mathrm{X}, \mathrm{T}, \theta, \mathrm{n}$ key, then the **MATH** button on your calculator:

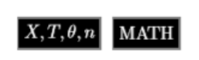


Figure 4.2.8).

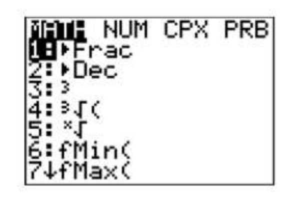
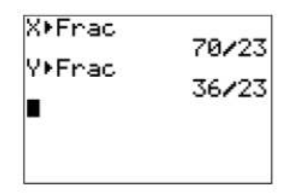


Figure 4.2.8: The **MATH** menu.

Select **1:▸Frac**, then press the **ENTER** key to produce the fractional equivalent of the decimal content of the variable X (see Figure 4.2.9).



```
X▸Frac
              70/23
Y▸Frac
              36/23
```

Figure 4.2.9: Changing the contents of the variables X and Y to fractions.

Repeat the procedure for the variable Y. Enter:

ALPHA 1 MATH

Figure 4.2.9). Note that the fractional equivalents for X and Y are $70/23$ and $36/23$, precisely the same answers we got with the substitution method above.

Exercise 4.2.3

Solve the following system of equations:

$$\begin{aligned} 3x - 5y &= 3 \\ 5x - 6y &= 2 \end{aligned}$$

Answer

$(-8/7, -9/7)$

Exceptional Cases Revisited

It is entirely possible that you might apply the substitution method to a system of equations that either have an infinite number of solutions or no solutions at all. Let's see what happens should you do that.

Example 4.2.4

Solve the following system of equations:

$$2x + 3y = 6 \tag{4.2.7}$$

$$y = -\frac{2}{3}x + 4 \tag{4.2.8}$$

Solution

Equation 4.2.8 is already solved for y, so let's substitute $-\frac{2}{3}x + 4$ for y in Equation 4.2.7.

$$\begin{aligned} 2x + 3y &= 6 && \text{Equation 4.2.7} \\ 2x + 3\left(-\frac{2}{3}x + 4\right) &= 6 && \text{Substitute } -\frac{2}{3}x + 4 \text{ for } y \\ 2x - 2x + 12 &= 6 && \text{Distribute the 3} \\ 12 &= 6 && \text{Simplify.} \end{aligned}$$

Goodness! What happened to the x? How are we supposed to solve for x in this situation? However, note that the resulting statement, $12 = 6$, is false, no matter what we use for x and y. This should give us a clue that there are no solutions. Perhaps we are dealing with parallel lines?

Let's solve Equation 4.2.7 for y, putting the equation into slope-intercept form, to help determine the situation.

$$\begin{aligned} 2x+3y &= 6 && \text{Equation 4.2.7} \\ 3y &= -2x+6 && \text{Subtract } 2x \text{ from both sides.} \\ y &= -\frac{2}{3}x+2 && \text{Divide both sides by 3} \end{aligned}$$

Thus, our system is equivalent to the following two equations.

$$\begin{aligned} y &= -\frac{2}{3}x+2 \\ y &= -\frac{2}{3}x+4 \end{aligned}$$

These lines have the same slope $-2/3$, but different y-intercepts (one has y-intercept $(0,2)$, the other has y-intercept $(0,4)$). Hence, these are two distinct parallel lines and the system has no solution.

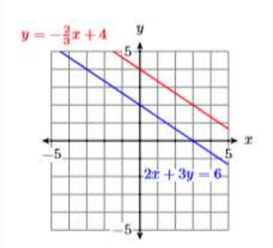


Figure 4.2.10: $2x+3y=6$ and $y=-\frac{2}{3}x+4$ are parallel. No solution.

Exercise 4.2.4

Solve the following system of equations:

$$\begin{aligned} x &= \frac{4}{3}y-7 \\ 6x-8y &= -3 \end{aligned}$$

Answer

no solution

Example 4.2.5

Solve the following system of equations:

$$2x-6y=-8 \tag{4.2.9}$$

$$x=3y-4 \tag{4.2.10}$$

Solution

Equation 4.2.10 is already solved for x, so let's substitute $3y-4$ for x in Equation 4.2.9.

$$\begin{aligned} 2x-6y &= -8 && \text{Equation 4.2.9} \\ 2(3y-4)-6y &= -8 && \text{Substitute } 3y-4 \text{ for } x \\ 6y-8-6y &= -8 && \text{Distribute the 2} \\ -8 &= -8 && \text{Simplify.} \end{aligned}$$

Goodness! What happened to the x? How are we supposed to solve for x in this situation? However, note that the resulting statement, $-8=-8$, is a true statement this time. Perhaps this is an indication that we are dealing with the same line? Let's put both equations 4.2.9 and 4.2.10 into slope-intercept form so that we can compare them.

Solve Equation 4.2.9 for y:

$$\begin{aligned} 2x - 6y &= -8 \\ -6y &= -2x - 8 \\ y &= \frac{-2x-8}{-6} \\ y &= \frac{1}{3}x + \frac{4}{3} \end{aligned}$$

Solve Equation 4.2.10 for y:

$$\begin{aligned} x &= 3y - 4 \\ x + 4 &= 3y \\ \frac{x+4}{3} &= y \\ y &= \frac{1}{3}x + \frac{4}{3} \end{aligned}$$

Hence, the lines have the same slope and the same y-intercept and they are exactly the same lines. Thus, there are an infinite number of solutions. Indeed, any point on either line is a solution. Examples of solution points are $(-4,0)$, $(-1,1)$, and $(2,2)$.

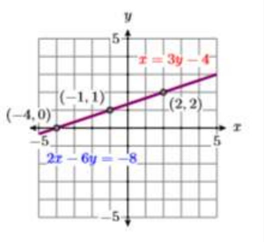


Figure 4.2.1: $2x - 6y = -8$ and $x = 3y - 4$ are the same line. Infinite number of solutions.

Exercise 4.2.5

Solve the following system of equations:

$$\begin{aligned} -28x + 14y &= -126 \\ y &= 2x - 9 \end{aligned}$$

Answer

There are an infinite number of solutions. Examples of solution points are $(0,-9)$, $(5,1)$, and $(-3,-15)$.

Tip

When you substitute one equation into another and the variable disappears, consider:

1. If the resulting statement is false, then you have two distinct parallel lines and there is no solution.
2. If the resulting statement is true, then you have the same lines and there are an infinite number of solutions.

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4.3: Solving Systems by Elimination

When both equations of a system are in standard form $Ax + By = C$, then a process called elimination is usually the best procedure to use to find the solution of the system. Elimination is based on two simple ideas, the first of which should be familiar.

1. Multiplying both sides of an equation by a non-zero number does not change its solutions. Thus, the equation

   $$x + 3y = 7 \tag{4.3.1}$$

   will have the same solutions (it's the same line) as the equation obtained by multiplying Equation 4.3.1 by 2:

   $$2x + 6y = 14 \tag{4.3.2}$$

2. Adding two true equations produces another true equation. For example, consider what happens when you add $4 = 4$ to $5 = 5$.

$$\begin{array}{c} 4 = 4 \\ 5 = 5 \\ \hline 9 = 9 \end{array}$$

Even more importantly, consider what happens when you add two equations that have $(2, 1)$ as a solution. The result is a third equation whose graph also passes through the solution.

$$\begin{array}{rcl} x + y & = & 3 \\ x - y & = & 1 \\ \hline 2x & = & 4 \\ x & = & 2 \end{array}$$

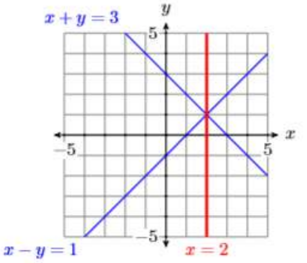


Figure $4.3.1$

fact

Adding a multiple of an equation to a second equation produces an equation that passes through the same solution as the first two equations.

One more important thing to notice is the fact that when we added the equations

$$\begin{array}{r} x + y = 3 \\ x - y = 1 \\ \hline 2x \quad = 4 \end{array}$$

the variable y was eliminated. This is where the elimination method gets its name. The strategy is to somehow add the equations of a system with the intent of eliminating one of the unknown variables. However, sometimes you need to do a little bit more than simply add the equations. Let's look at an example.

Example $4.3.1$

Solve the following system of equations:

$$x + 2y = -5 \tag{4.3.3}$$

$$2x - y = -5 \tag{4.3.4}$$

Solution

Our focus will be on eliminating the variable x. Note that if we multiply Equation 4.3.3 by -2, then add the result to Equation 4.3.4, the x terms will be eliminated.

$$\begin{aligned} -2x-4y&=10 \quad \text{Multiply equation4.3.3 by } -2 \\ 2x-y&=-5 \quad \text{Multiply equation4.3.4} \\ \hline -5y&=5 \quad \text{Add the equations.} \end{aligned}$$

Divide both sides of $-5y=5$ by -5 to get $y=-1$. To find the corresponding value of x, substitute -1 for y in Equation 4.3.3 (or Equation 4.3.4) and solve for x.

$$\begin{aligned} x+2y&=-5 \quad \text{Equation 4.3.3} \\ x+2(-1)&=-5 \quad \text{Substitute } -1 \text{ for } y \\ x&=-3 \quad \text{Solve for } x. \end{aligned}$$

Check: To check, we need to show that the point $(x,y)=(-3,1)$ satisfies both equations.

Substitute $(x,y)=(-3,-1)$ into Equation 4.3.3.

$$\begin{aligned} x+2y&=-5 \\ -3+2(-1)&=-5 \\ -5&=-5 \end{aligned}$$

Substitute $(x,y)=(-3,-1)$ intoEquation 4.3.4.

$$\begin{aligned} 2x-y&=-5 \\ 2(-3)-(-1)&=-5 \\ -5&=-5 \end{aligned}$$

Thus, the point $(x,y)=(-3,-1)$ satisfies both equations and is therefore the solution of the system.

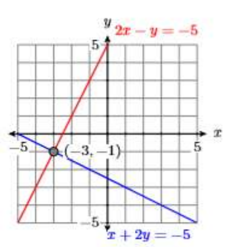


Figure 4.3.2: $x+2y=-5$ and $2x-y=-5$ intersect at $(-3,-1)$.

Exercise 4.3.1

Solve the following system of equations:

$$\begin{aligned} x+3y&=14 \\ -8x-3y&=-28 \end{aligned}$$

Answer

$(2,4)$

To show that you have the option of which variable you choose to eliminate, let's try Example 4.3.1 a second time, this time eliminating y instead of x.

Example 4.3.2

Solve the following system of equations:

$$x+2y=-5 \tag{4.3.5}$$

$$2x-y=-5 \tag{4.3.6}$$

Solution

This time we focus on eliminating the variable y. We note that if we multiply Equation 4.3.6 by 2, then add the result to Equation 4.3.5, the y terms will be eliminated.

$$\begin{array}{rl} x+2y=-5 & \text{Equation 4.3.5} \\ 4x-2y=-10 & \text{Multiple equation 4.3.6 by 2} \\ \hline 5x \qquad =-15 & \text{Add the equations.} \end{array}$$

Divide both sides of $5x=-15$ by 5 to get $x=-3$. To find the corresponding value of y, substitute -3 for x in Equation 4.3.5 (or Equation 4.3.6) and solve for y.

$$\begin{aligned} x+2y&=-5 && \text{Equation 4.3.5} \\ -3+2y&=-5 && \text{Substitute } -3 \text{ for } x \\ 2y&=-2 && \text{Add 3 to both sides.} \\ y&=-1 && \text{Divide both sides by 2} \end{aligned}$$

Hence, $(x, y)=(-3,-1)$, just as in Example $4.3.1$, is the solution of the system.

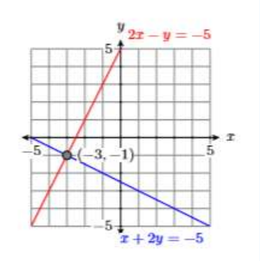


Figure $4.3.3$: $x+2y=-5$ and $2x-y=-5$ intersect at $(-3,-1)$.

Sometimes elimination requires a thought process similar to that of finding a common denominator.

Example $4.3.3$

Solve the following system of equations:

$$3x+4y=12 \tag{4.3.7}$$

$$2x-5y=10 \tag{4.3.8}$$

Solution

Let's focus on eliminating the x-terms. Note that if we multiply Equation 4.3.7 by 2, then multiply Equation 4.3.8 by -3, the x-terms will be eliminated when we add the resulting equations.

$$\begin{array}{rl} 6x+8y=24 & \text{Multiple equation 4.3.7 by 2} \\ -6x+15y=-30 & \text{Multiple equation 4.3.8 by 2} \\ \hline 23y=-6 & \text{Add the equations.} \end{array}$$

Hence, $y=-6/23$.

At this point, we could substitute $y=-6/23$ in either equation, then solve the result for x. However, working with $y=-6/23$ is a bit daunting, particularly in the light of elimination being easier. So let's use elimination again, this time focusing on eliminating y. Note that if we multiply Equation 4.3.7 by 5, then multiply Equation 4.3.8 by 4, when we add the results, the y-terms will be eliminated.

$$\begin{array}{rl} 5x+20y=60 & \text{Multiple equation 4.3.7 by 5} \\ 8x-20y=40 & \text{Multiple equation 4.3.8 by 4} \\ \hline 23x \qquad =100 & \text{Add the equations.} \end{array}$$

Thus, $x=100/23$, and the system of the system is $(x, y)=(100/23,-6/23)$.

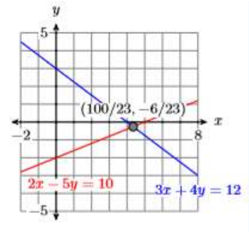
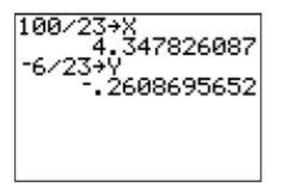


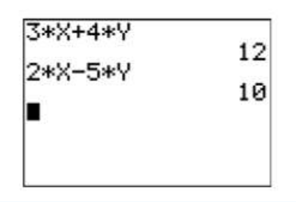
Figure $4.3.4$: $3x+4y=12$ and $2x-5y=10$ intersect at $(100/23,-6/23)$.

Check: Let's use the graphing calculator to check the solution. First, store $100/23$ in X, then $-6/23$ in Y (see Figure $4.3.5$). Next, enter the left-hand sides of equations 4.3.7 and 4.3.8.



```
100/23→X
    4.347826087
-6/23→Y
   -.2608695652
```

Figure $4.3.5$: Enter $100/23$ in X, $-6/23$ in Y.



```
3*X+4*Y
             12
2*X-5*Y
             10
■
```

Figure $4.3.6$: Enter the left-hand sides of equations 4.3.7 and 4.3.8.

Note that both calculations in Figure $4.3.6$ provide the correct right-hand sides for equations 4.3.7 and 4.3.8. Thus, the solution $(x,y)=(100/23,-6/23)$ checks.

Exercise $4.3.3$

Solve the following system of equations:

$$\begin{aligned}-14x+9y&=94\\7x+3y&=-62\end{aligned}$$

Answer

$(-8,-2)$

Exceptional Cases

In the previous section, we saw that if the substitution method led to a false statement, then we have parallel lines. The same thing can happen with the elimination method of this section.

Example $4.3.4$

Solve the following system of equations:

$$x+y=3 \tag{4.3.9}$$

$$2x+2y=6 \tag{4.3.10}$$

Solution

Let's focus on eliminating the x-terms. Note that if we multiply Equation 4.3.9 by -2, the x-terms will be eliminated when we add the resulting equations.

$$-2x-2y=-6 \quad \text{Multiple equation 4.3.9 by } -2$$

$$\begin{array}{rl} 2x \quad 2y \quad\; 6 & \text{Multiple equation 4.3.9 by } \quad 2 \\ 2x+2y=-6 & \text{Equation 4.3.10} \\ \hline 0=-12 & \text{Add the equations.} \end{array}$$

Because of our experience with this solving this exceptional case with substitution, the fact that both variables have disappeared should not be completely surprising. Note that this last statement is false, regardless of the values of x and y. Hence, the system has no solution.

Indeed, if we find the intercepts of each equation and plot them, then we can easily see that the lines of this system are parallel (see Figure 4.3.7). Parallel lines never intersect, so the system has no solutions.

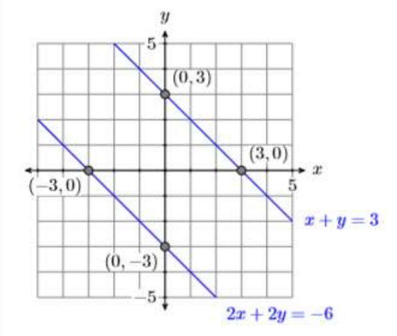


Figure 4.3.7: The lines $x+y=3$ and $2x+2y=-6$ are parallel.

Exercise 4.3.4

Solve the following system of equations:

$$\begin{aligned} 5x-4y&=-16 \\ 15x-12y&=49 \end{aligned}$$

Answer

no solution

In the previous section, we saw that if the substitution method led to a true statement, then we have the same lines. The same thing can happen with the elimination method of this section.

Example 4.3.5

Solve the following system of equations:

$$x-7y=4 \tag{4.3.11}$$

$$-3x+21y=-12 \tag{4.3.12}$$

Solution

If we are not on automatic pilot late at night doing our homework, we might recognize that the equations 4.3.11 and 4.3.12 are identical. But it's also conceivable that we don't see that right away and begin the elimination method. Let's multiply the first equation by 3, then add. This will eliminate the x-terms.

$$\begin{array}{rl} 3x-21y=12 & \text{Multiple equation 4.3.11 by 3} \\ 3x+21y=-12 & \text{Equation 4.3.12} \\ \hline 0=0 & \text{Add the equations.} \end{array}$$

Again, all of the variables have disappeared! However, this time the last statement is true, regardless of the values of x and y.

Notice that if we multiply Equation 4.3.11 by -3, then we have two identical equations.

$$\begin{array}{rl} -3x+21y=-12 & \text{Multiple equation 4.3.11 by 3} \\ -3x+21y=-12 & \text{Equation 4.3.12} \end{array}$$

The equations are identical! Hence, there are an infinite number of points of intersection. Indeed, any point on either line is a solution. Example points of solution are $(-3,-1)$, $(0,-4/7)$, and $(4,0)$.

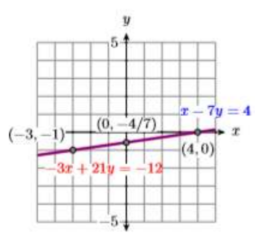


Figure 4.3.8: $x-7y=4$ and $-3x+21y=-12$ are the same line. Infinite number of solutions.

Exercise 4.3.5

Solve the following system of equations:

$$\begin{aligned} 2x-7y&=4\\ 8x-28y&=16 \end{aligned}$$

Answer

There are an infinite number of solutions. Examples of solution points are $(2,0)$, $(9,2)$, and $(-5,-2)$.

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4.4: Applications of Linear Systems

In this section we create and solve applications that lead to systems of linear equations. As we create and solve our models, we'll follow the Requirements for Word Problem Solutions from Chapter 2, Section 5. However, instead of setting up a single equation, we set up a system of equations for each application.

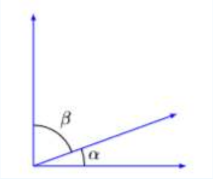
Example 4.4.1

In geometry, two angles that sum to 90° are called complementary angles. If the second of two complementary angles is 30° larger than twice the first angle, find the degree measure of both angles.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Our variable dictionary will take the form of a diagram, naming the two complementary angles α and β.



2. *Set up a Systems of Equations*. The "second angle is 30 degrees larger than twice the first angle" becomes

$$\beta=30+2\alpha \tag{4.4.1}$$

Secondly, the angles are complementary, meaning that the sum of the angles is 90°.

$$\alpha+\beta=90 \tag{4.4.2}$$

Thus, we have a system of two equations in two unknowns α and β.

3. *Solve the System*. As Equation 4.4.1 is already solved for β, let use the substitution method and substitute $30+2\alpha$ for β in Equation 4.4.2.

$$\begin{aligned} \alpha+\beta &= 90 && \text{Equation 4.4.2} \\ \alpha+(30+2\alpha) &= 90 && \text{Substitute } 30+2\alpha \text{ for } \beta \\ 3\alpha+30 &= 90 && \text{Combine like terms.} \\ 3\alpha &= 60 && \text{Subtract 30 from both sides.} \\ \alpha &= 20 && \text{Divide both sides by 3} \end{aligned}$$

4. *Answer the Question*. The first angle is $\alpha=20$ degrees. The second angle is:

$$\begin{aligned} &\beta=30+2\alpha \quad \text{Equation 4.4.1} \\ &\beta=30+2(20) \quad \text{Substitute 20 for } \alpha \\ &\beta=70 \quad \text{Simplify.} \end{aligned}$$

5. *Look Back*. Certainly 70° is 30° larger than twice 20°. Also, note that $20^{\circ}+70^{\circ}=90^{\circ}$, so the angles are complementary. We have the correct solution.

Exercise 4.4.1

If the second of two complementary angles is 6° larger than 3 times the first angle, find the degree measure of both angles.

Answer

21 and 69

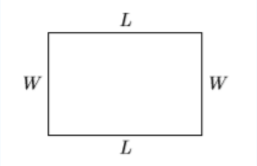
Example 4.4.2

The perimeter of a rectangle is 280 feet. The length of the rectangle is 10 feet less than twice the width. Find the width and length of the rectangle.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Our variable dictionary will take the form of a diagram, naming the width and length W and L, respectively.



2. *Set up a System of Equations*. The perimeter is found by summing the four sides of the rectangle.

$$\begin{aligned} P &= L+W+L+W \\ P &= 2L+2W \end{aligned}$$

We're told the perimeter is 280 feet, so we can substitute 280 for P in the last equation.

$$280 = 2L+2W$$

We can simplify this equation by dividing both sides by 2, giving the following result:

$$L+W=140$$

Secondly, we're told that the "length is 10 feet less than twice the width." This translates to:

$$L=2W-10$$

Thus, the system we need to solve is:

$$L+W=140 \tag{4.4.3}$$

$$L=2W-10 \tag{4.4.4}$$

3. *Solve the System*. As Equation 4.4.4 is already solved for L, let use the substitution method and substitute $2W-10$ for L in Equation 4.4.3.

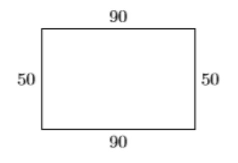
$$\begin{aligned} W+L &= 140 && \text{Equation 4.4.3} \\ W+(2W-10) &= 140 && \text{Substitute } 2W-10 \text{ for } L \\ 3W-10 &= 140 && \text{Combine like terms.} \\ 3W &= 150 && \text{Add 10 to both sides.} \\ W &= 50 && \text{Divide both sides by 3} \end{aligned}$$

4. *Answer the Question*. The width is $W=50$ feet. The length is:

$$\begin{aligned} L &= 2W-10 && \text{Equation 4.4.4} \\ L &= 2(50)-10 && \text{Substitute 50 for } W. \\ L &= 90 && \text{Simplify.} \end{aligned}$$

Thus, the length is $L=90$ feet.

5. *Look Back*. Perhaps a picture, labeled with our answers might best demonstrate that we have the correct solution. Remember, we found that the width was 50 feet and the length was 90 feet.



Note that the perimeter is $P = 90 + 50 + 90 + 50 = 280$ feet. Secondly, note that the length 90 feet is 10 feet less than twice the width. So we have the correct solution.

Exercise 4.4.2

The perimeter of a rectangle is 368 meters. The length of the rectangle is 34 meters more than twice the width. Find the width and length of the rectangle.

Answer

length $= 134$, width $= 50$

Example 4.4.3

Pascal has $\$3.25$ in change in his pocket, all in dimes and quarters. He has 22 coins in all. How many dimes does he have?

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let D represent the number of dimes and let Q represent the number of quarters.
2. *Set up a System of Equations*. Using a table to summarize information is a good strategy. In the first column, we list the type of coin. The second column gives the number of each type of coin, and the third column contains the value (in cents) of the number of coins in Pascal's pocket.

	Number of Coins	Value (in cents)
Dimes	D	$10D$
Quarters	Q	$25Q$
Totals	22	325

Note that D times, valued at 10 cents apiece, are worth $10D$ cents. Similarly, Q quarters, valued at 25 cents apiece, are worth $25Q$ cents. Note also how we've change $\$3.25$ to 325 cents. The second column of the table gives us our first equation.

$$D + Q = 22 \tag{4.4.5}$$

The third column of the table gives us our second equation.

$$10D + 25Q = 325 \tag{4.4.6}$$

3. *Solve the System*. Because equations 4.4.5 and 4.4.6 are both in standard form $Ax + By = C$, we'll use the elimination method to find a solution. Because the question asks us to find the number of dimes in Pascal's pocket, we'll focus on eliminating the Q-terms and keeping the D-terms.

$$\begin{array}{rcll} -25D - 25Q &=& -550 & \text{Multiply equation 4.4.5 by } -25 \\ 10D + 25Q &=& 325 & \text{Equation 4.4.6} \\ \hline -15D \quad\quad &=& -225 & \text{Add the equations.} \end{array}$$

Dividing both sides of the last equation by -15 gives us $D = 15$.

4. *Answer the Question*. The previous solution tells us that Pascal has 15 dimes in his pocket.
5. *Look Back*. Again, summarizing results in a table might help us see if we have the correct solution. First, because we're told that Pascal has 22 coins in all, and we found that he had 15 dimes, this means that he must have 7 quarters.

	Number of Coins	Value (in cents)
Dimes	15	150

	Number of Coins	Value (in cents)
Quarters	7	175
Totals	22	325

Fifteen dimes are worth 150 cents, and 7 quarters are worth 175 cents. That's a total of 22 coins and 325 cents, or $\$3.25$. Thus we have the correct solution.

Exercise $4.4.3$

Eloise has $\$7.10$ in change in her pocket, all in nickels and quarters. she has 46 coins in all. How many quarters does she have?

Answer

24

Example $4.4.4$

Rosa inherits $\$10,000$ and decides to invest the money in two accounts, one portion in a certificate of deposit that pays 4% interest per year, and the rest in a mutual fund that pays 5% per year. At the end of the first year, Rosa's investments earn a total of $\$420$ in interest. Find the amount invested in each account.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let C represent the amount invested in the certificate of deposit and M represent the amount invested in the mutual fund.
2. *Set up a System of Equations*. We'll again use a table to summarize information.

	Rate	Amount invested	Interest
Certificate of Deposit	4%	C	$0.04C$
Mutual Fund	5%	M	$0.05M$
Totals		$10,000$	420

At 4%, the interest earned on a C dollars investment is found by taking 4% of C (i.e., $0.04C$). Similarly, the interest earned on the mutual fund is $0.05M$. The third column of the table gives us our first equation. The total investment is $\$10,000$.

$$C + M = 10000$$

The fourth column of the table gives us our second equation. The total interest earned is the sum of the interest earned in each account.

$$0.04C + 0.05M = 420$$

Let's clear the decimals from the last equation by multiplying both sides of the equation by 100.

$$4C + 5M = 42000$$

Thus, the system we need to solve is:

$$C+M=10000 \tag{4.4.7}$$

$$4C+5M=42000 \tag{4.4.8}$$

3. *Solve the System*. Because equations 4.4.7 and 4.4.8 are both in standard form $Ax+By=C$, we'll use the elimination method to find a solution. We'll focus on eliminating the C-terms.

$$\begin{array}{rrl} -4C-4M= & -40000 & \text{Multiply equation 4.4.7 by } -4 \\ 4C+5M= & 42000 & \text{Equation 4.4.8} \\ \hline -15M= & 2000 & \text{Add the equations.} \end{array}$$

Thus, the amount invested in the mutual fund in $M=\$2,000$.

4. *Answer the Question*. The question asks us to find the amount invested in each account. So, substitute $M=2000$ in Equation 4.4.7 and solve for C.

$$\begin{array}{rrl} C+M= & 10000 & \text{Equation 4.4.7} \\ C+2000= & 10000 & \text{Substitute 2000 for M} \\ C= & 8000 & \text{Subtract 2000 from both sides.} \end{array}$$

Thus $C=\$8,000$ was invested in the certificate of deposit.

5. *Look Back*. First, note that the investments in the certificate of deposit and the mutual fund, $\$8,000$ and $\$2,000$ respectively, total $\$10,000$. Let's calculate the interest on each investment: 4% of $\$8,000$ is $\$320$ and 5% of $\$2,000$ is $\$100$.

	Rate	Amount invested	Interest
Certificate of Deposit	4%	$8,000$	320
Mutual Fund	5%	$2,000$	100
Totals		$10,000$	420

Note that the total interest is $\$420$, as required in the problem statement. Thus, our solution is correct.

Exercise 4.4.4

Eileen inherits $\$40,000$ and decides to invest the money in two accounts, part in a certificate of deposit that pays 3% interest per year, and the rest in a mutual fund that pays 6% per year. At the end of the first year, her investments earn a total of $\$2,010$ in interest. Find the amount invested in each account.

Answer

$\$13,000$ in the certificate of deposit, $\$27,000$ in the mutual fund.

Example 4.4.5

Peanuts retail at $\$0.50$ per pound and cashews cost $\$1.25$ per pound. If you were a shop owner, how many pounds of peanuts and cashews should you mix to make 50 pounds of a peanut-cashew mixture costing $\$0.95$ per pound?

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary*. Let P be the number of pounds of peanuts used and let C be the number of pounds of cashews used.
2. *Set up a System of Equations*. We'll again use a table to summarize information.

	Cost per pound	Amount (pounds)	Cost
Peanuts	$\$0.50$	P	$0.50P$
Cashews	$\$1.25$	C	$1.25C$
Totals	$\$0.95$	50	$0.95(50)=47.50$

At $\$0.50$ per pound, P pounds of peanuts cost $0.50P$. At $\$1.25$ per pound, C pounds of cashews cost $1.25C$. Finally, at $\$0.95$ per pound, 50 pounds of a mixture of peanuts and cashews will cost $0.95(50)$, or $\$47.50$. The third column of the table gives us our first equation. The total number of pounds of mixture is given by the following equation:

$$P+C=50$$

The fourth column of the table gives us our second equation. The total cost is the sum of the costs for purchasing the peanuts and cashews.

$$0.50P+1.25C=47.50$$

Let’s clear the decimals from the last equation by multiplying both sides of the equation by 100.

$$50P+125C=4750$$

Thus, the system we need to solve is:

$$P+C=50 \tag{4.4.9}$$

$$50P+125C=4750 \tag{4.4.10}$$

3. *Solve the System*. Because equations 4.4.9 and 4.4.10 are both in standard form $Ax+By=C$, we’ll use the elimination method to find a solution. We’ll focus on eliminating the P-terms.

$$\begin{array}{rrl} -50P-50C= & -2500 & \text{Multiply equation 4.4.9 by } -50 \\ 50P+125C= & 4750 & \text{Equation 4.4.10} \\ \hline 75C= & 2250 & \text{Add the equations.} \end{array}$$

Divide both sides by 75 to get $C=30$ pounds of cashews are in the mix.

4. *Answer the Question*. The question asks for both amounts, peanuts and cashews. Substitute $C=30$ in Equation 4.4.9 to determine P.

$$\begin{array}{rl} P+C=50 & \text{Equation 4.4.9} \\ C+30=50 & \text{Substitute 30 for C} \\ P=20 & \text{Subtract 30 from both sides.} \end{array}$$

Thus, there are $P=20$ pounds of peanuts in the mix.

5. *Look Back*. First, note that the amount of peanuts and cashews in the mix is 20 and 30 pounds respectively, so the total mixture weighs 50 pounds as required. Let’s calculate the costs: for the peanuts, $0.50(20)$, or $\$10$, for the cashews, $1.25(30)=37.50$.

	Cost per pound	Amount (pounds)	Cost
Peanuts	$0.50	20	$10.00
Cashews	$1.25	30	$37.50
Totals	$0.95	50	$47.50

Note that the total cost is $\$47.50$, as required in the problem statement. Thus, our solution is correct.

Exercise $4.4.5$

A store sells peanuts for $\$4.00$ per pound and pecans for $\$7.00$ per pound. How many pounds of peanuts and how many pounds of pecans should you mix to make a 25-lb mixture costing $\$5.80$ per pound?

Answer

10 pounds of peanuts, 15 pounds of pecans

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4.E: Systems of Linear Equations (Exercises)

4.1: Solving Systems by Graphing

In Exercises 1-6, solve each of the given systems by sketching the lines represented by each equation in the system, then determining the coordinates of the point of intersection. Each of these problems have been designed so that the coordinates of the intersection point are integers. Check your solution.

1) $$\begin{aligned} 3x-4y&=24\\ y&=-\frac{1}{2}x-1 \end{aligned}$$

Answer

$(4,-3)$

2) $$\begin{aligned} x-4y&=-8\\ y&=\frac{5}{4}x+6 \end{aligned}$$

3) $$\begin{aligned} 2x+y&=6\\ y&=x+3 \end{aligned}$$

Answer

$(1,4)$

4) $$\begin{aligned} x-2y&=-4\\ y&=\frac{5}{2}x+6 \end{aligned}$$

5) $$\begin{aligned} x+2y&=-6\\ y&=-3x-8 \end{aligned}$$

Answer

$(-2,-2)$

6) $$\begin{aligned} x-3y&=6\\ y&=2x-7 \end{aligned}$$

In Exercises 7-18, solve each of the given systems by sketching the lines represented by each equation of the given system on graph paper, then estimating the coordinates of the point of intersection to the nearest tenth. Check the solution.

7) $$\begin{aligned} x-3y&=3\\ x-4y&=-4 \end{aligned}$$

Answer

$(-3.4,0.1)$

8) $$\begin{aligned} 4x-3y&=-12\\ -x-4y&=4 \end{aligned}$$

9) $$\begin{aligned} -3x+3y&=-9\\ -3x+3y&=-12 \end{aligned}$$

Answer

No solution. Lines are parallel.

10) $$\begin{aligned} x-y&=-2\\ 2x-2y&=6 \end{aligned}$$

11) $$\begin{aligned} 6x-7y&=-42\\ y&=-\frac{1}{4}x+4 \end{aligned}$$

Answer

$(-1.8, 4.5)$

12) $\begin{aligned} 4x+3y&=24 \\ y&=\frac{1}{7}x+5 \end{aligned}$

13) $\begin{aligned} 6x-7y&=-42 \\ y&=-\frac{1}{5}x+2 \end{aligned}$

Answer

$(-3.8, 2.8)$

14) $\begin{aligned} 7x-8y&=56 \\ y&=-\frac{1}{3}x-4 \end{aligned}$

15) $\begin{aligned} 6x+3y&=12 \\ -2x-y&=4 \end{aligned}$

Answer

No solution. Lines are parallel.

16) $\begin{aligned} x-4y&=-4 \\ -x+4y&=-4 \end{aligned}$

17) $\begin{aligned} 3x+y&=3 \\ -2x+3y&=-6 \end{aligned}$

Answer

$(1.4, -1.1)$

18) $\begin{aligned} 2x-y&=-2 \\ -x-2y&=4 \end{aligned}$

In Exercises 19-24, use the graphing calculator to solve the given system. Round your answer to the nearest tenth. Use the Calculator Submission Guidelines from Chapter 3, Section 2, when reporting your answer on your homework.

19) $\begin{aligned} y&=\frac{3}{4}x+7 \\ y&=-\frac{1}{3}x+2 \end{aligned}$

Answer

$(-4.6, 3.5)$

20) $\begin{aligned} y&=\frac{7}{6}x+6 \\ y&=-\frac{1}{7}x+3 \end{aligned}$

21) $\begin{aligned} y&=\frac{4}{3}x-3 \\ y&=-\frac{4}{7}x-1 \end{aligned}$

Answer

$(1.1, -1.6)$

22) $\begin{aligned} y&=\frac{8}{3}x-3 \\ y&=-\frac{1}{8}x-2 \end{aligned}$

23) $\begin{aligned} y&=\frac{1}{6}x+1 \\ y&=-\frac{3}{7}x+5 \end{aligned}$

Answer

$(6.7, 2.1)$

24) $\begin{aligned} y &= \frac{5}{8}x + 3 \\ y &= -\frac{5}{6}x - 5 \end{aligned}$

In Exercises 25-30, use the graphing calculator to solve the given system. Round your answer to the nearest tenth. Use the Calculator Submission Guidelines when reporting your answer on your homework.

25) $\begin{aligned} 6x + 16y &= 96 \\ -6x + 13y &= -78 \end{aligned}$

Answer

$(14.3, 0.6)$

26) $\begin{aligned} -4x + 16y &= -64 \\ 5x + 8y &= 40 \end{aligned}$

27) $\begin{aligned} -2x - 11y &= 22 \\ 8x - 12y &= -96 \end{aligned}$

Answer

$(-11.8, 0.1)$

28) $\begin{aligned} -6x - 10y &= 60 \\ 2x - 18y &= -36 \end{aligned}$

29) $\begin{aligned} -6x + 2y &= -12 \\ -12x + 3y &= -36 \end{aligned}$

Answer

$(6.0, 12.0)$

30) $\begin{aligned} -3x + y &= -3 \\ -14x + 3y &= -42 \end{aligned}$

4.2: Solving Systems by Substitution

In Exercises 1-8, use the substitution method to solve each of the following systems. Check your answer manually, without the use of a calculator.

1) $\begin{aligned} -7x + 7y &= 63 \\ y &= 6 - 2x \end{aligned}$

Answer

$(-1, 8)$

2) $\begin{aligned} 3x - 8y &= 27 \\ y &= 4 - 7x \end{aligned}$

3) $\begin{aligned} x &= 19 + 7y \\ 3x - 3y &= 3 \end{aligned}$

Answer

$(3, -4)$

4) $\begin{aligned} x &= 39 + 8y \\ -9x + 2y &= -71 \end{aligned}$

5) $\begin{aligned} x &= -5 - 2y \\ -2x - 6y &= 18 \end{aligned}$

Answer

$(3,-4)$

6) $\begin{aligned} x &= 15+6y \\ 9x+3y &= 21 \end{aligned}$

7) $\begin{aligned} 6x-8y &= 24 \\ y &= 15+3x \end{aligned}$

Answer

$(-8,-9)$

8) $\begin{aligned} 9x+8y &= -45 \\ y &= 15-8x \end{aligned}$

In Exercises 9-28, use the substitution method to solve each of the following systems.

9) $\begin{aligned} -x+9y &= 46 \\ 7x-4y &= -27 \end{aligned}$

Answer

$(-1,5)$

10) $\begin{aligned} -x+9y &= -12 \\ 4x+7y &= -38 \end{aligned}$

11) $\begin{aligned} -x+4y &= 22 \\ 8x+7y &= -20 \end{aligned}$

Answer

$(-6,4)$

12) $\begin{aligned} -x-2y &= 15 \\ 3x-9y &= 15 \end{aligned}$

13) $\begin{aligned} x+2y &= -4 \\ 6x-4y &= -56 \end{aligned}$

Answer

$(-8,2)$

14) $\begin{aligned} x+8y &= 79 \\ 4x+6y &= 8 \end{aligned}$

15) $\begin{aligned} x+6y &= -49 \\ -3x+4y &= -7 \end{aligned}$

Answer

$(-7,-7)$

16) $\begin{aligned} x-4y &= 33 \\ 4x+7y &= -6 \end{aligned}$

17) $\begin{aligned} -2x+8y &= -50 \\ -9x-y &= -3 \end{aligned}$

Answer

$(1,-6)$

18) $\begin{aligned} -6x-6y &= 102 \\ 9x-y &= -63 \end{aligned}$

19) $\begin{aligned} -4x-8y &= -4 \\ y &= -2y-4 \end{aligned}$

Answer

No solution

20) $\begin{aligned} 3x+6y&=2 \\ y&=-2y+2 \end{aligned}$

21) $\begin{aligned} -2x-2y&=26 \\ -7x+y&=19 \end{aligned}$

Answer

$(-4,-9)$

22) $\begin{aligned} -2x-8y&=-30 \\ -6x+y&=10 \end{aligned}$

23) $\begin{aligned} 3x-4y&=-43 \\ -3x+y&=22 \end{aligned}$

Answer

$(-5,7)$

24) $\begin{aligned} -2x+8y&=14 \\ 8x+y&=43 \end{aligned}$

25) $\begin{aligned} -8x-4y&=24 \\ 9x-y&=-71 \end{aligned}$

Answer

$(-7,8)$

26) $\begin{aligned} -8x-2y&=-14 \\ -6x-y&=-9 \end{aligned}$

27) $\begin{aligned} -8x-7y&=2 \\ y&=-\frac{8}{7}x+9 \end{aligned}$

Answer

No solution

28) $\begin{aligned} 9x+4y&=-3 \\ y&=-\frac{9}{4}x+6 \end{aligned}$

In Exercises 29-36, use the substitution method to solve each of the following systems. Use your graphing calculator to check your solution.

29) $\begin{aligned} 3x-5y&=3 \\ 5x-7y&=2 \end{aligned}$

Answer

$(-11/4,-9/4)$

30) $\begin{aligned} 4x-5y&=4 \\ 3x-2y&=-1 \end{aligned}$

31) $\begin{aligned} 4x+3y&=8 \\ 3x+4y&=2 \end{aligned}$

Answer

$(26/7,-16/7)$

32) $\begin{aligned} 3x+8y&=3 \\ -4x-9y&=-2 \end{aligned}$

33) $\begin{aligned} 3x+8y&=6 \\ 2x+7y&=-2 \end{aligned}$

Answer

$(58/5, -18/5)$

34) $\begin{aligned} 3x-7y&=6 \\ 2x-3y&=1 \end{aligned}$

35) $\begin{aligned} 4x+5y&=4 \\ -3x-2y&=1 \end{aligned}$

Answer

$(-13/7, 16/7)$

36) $\begin{aligned} 5x+4y&=5 \\ 4x+5y&=2 \end{aligned}$

In Exercises 37-48, use the substitution method to determine how many solutions each of the following linear systems has.

37) $\begin{aligned} -9x+6y&=9 \\ y&=\frac{3}{2}x-8 \end{aligned}$

Answer

No solutions

38) $\begin{aligned} 3x-5y&=9 \\ y&=\frac{3}{5}x+6 \end{aligned}$

39) $\begin{aligned} y&=-2x-16 \\ -14x-7y&=112 \end{aligned}$

Answer

Infinite number of solutions

40) $\begin{aligned} y&=-12x+12 \\ 120x+10y&=120 \end{aligned}$

41) $\begin{aligned} x&=16-5y \\ -4x+2y&=24 \end{aligned}$

Answer

One solution

42) $\begin{aligned} x&=-18-4y \\ 7x-7y&=49 \end{aligned}$

43) $\begin{aligned} y&=7y+18 \\ 9x-63y&=162 \end{aligned}$

Answer

Infinite number of solutions

44) $\begin{aligned} y&=4y-9 \\ -10x+40y&=90 \end{aligned}$

45) $\begin{aligned} x&=-2y+3 \\ 4x+8y&=4 \end{aligned}$

Answer

No solutions

46) $\begin{aligned} x&=2y+4 \\ -3x+6y&=5 \end{aligned}$

47) $\begin{aligned} -9x+4y&=73 \\ y&=-3-2x \end{aligned}$

Answer

One solution

48)
$$\begin{aligned} 6x+9y&=27 \\ y&=16-5x \end{aligned}$$

4.3: Solving Systems by Elimination

In Exercises 1-8, use the elimination method to solve each of the following systems. Check your result manually, without the assistance of a calculator.

1)
$$\begin{aligned} x+4y&=0 \\ 9x-7y&=-43 \end{aligned}$$

Answer

$(-4, 1)$

2)
$$\begin{aligned} x+6y&=-53 \\ 5x-9y&=47 \end{aligned}$$

3)
$$\begin{aligned} 6x+y&=8 \\ 4x+2y&=0 \end{aligned}$$

Answer

$(2, -4)$

4)
$$\begin{aligned} 4x+y&=18 \\ -2x+6y&=-22 \end{aligned}$$

5)
$$\begin{aligned} -8x+y&=-56 \\ 4x+3y&=56 \end{aligned}$$

Answer

$(8, 8)$

6)
$$\begin{aligned} 2x+y&=21 \\ 7x+8y&=87 \end{aligned}$$

7)
$$\begin{aligned} x+8y&=41 \\ -5x-9y&=-50 \end{aligned}$$

Answer

$(1, 5)$

8)
$$\begin{aligned} x-4y&=-31 \\ -2x-6y&=-36 \end{aligned}$$

In Exercises 9-16, use the elimination method to solve each of the following systems.

9)
$$\begin{aligned} -12x+9y&=0 \\ -6x-4y&=-34 \end{aligned}$$

Answer

$(3, 4)$

10)
$$\begin{aligned} -27x-5y&=148 \\ -9x-3y&=60 \end{aligned}$$

11)
$$\begin{aligned} 27x-6y&=-96 \\ -3x-5y&=22 \end{aligned}$$

Answer

$(-4, -2)$

12)
$$\begin{aligned} -8x+8y&=-32 \\ 2x-9y&=15 \end{aligned}$$

13) $\begin{aligned} 2x-6y&=28 \\ -3x+18y&=-60 \end{aligned}$

Answer

$(8,-2)$

14) $\begin{aligned} -8x-6y&=96 \\ 4x+30y&=-156 \end{aligned}$

15) $\begin{aligned} -32x+7y&=-238 \\ 8x-4y&=64 \end{aligned}$

Answer

$(7,-2)$

16) $\begin{aligned} 12x+6y&=30 \\ -2x+7y&=51 \end{aligned}$

In Exercises 17-24, use the elimination method to solve each of the following systems.

17) $\begin{aligned} 3x-7y&=-75 \\ -2x-2y&=-10 \end{aligned}$

Answer

$(-4,9)$

18) $\begin{aligned} -8x+3y&=42 \\ -7x+8y&=26 \end{aligned}$

19) $\begin{aligned} 9x-9y&=-63 \\ 2x-6y&=-34 \end{aligned}$

Answer

$(-2,5)$

20) $\begin{aligned} -4x-8y&=-52 \\ -7x-3y&=-14 \end{aligned}$

21) $\begin{aligned} -9x-2y&=28 \\ 5x-3y&=-32 \end{aligned}$

Answer

$(-4,4)$

22) $\begin{aligned} -8x-2y&=-12 \\ 6x+3y&=12 \end{aligned}$

23) $\begin{aligned} -3x-5y&=-34 \\ 7x+7y&=56 \end{aligned}$

Answer

$(3,5)$

24) $\begin{aligned} -9x-9y&=9 \\ 7x+4y&=8 \end{aligned}$

In Exercises 25-32, use the elimination method to solve each of the following systems. Use your calculator to check your solutions.

25) $\begin{aligned} 2x-7y&=-2 \\ 7x+6y&=3 \end{aligned}$

Answer

$(9/61, 20/61)$

26) $\begin{aligned} -9x - 4y &= 4 \\ 5x - 3y &= -1 \end{aligned}$

27) $\begin{aligned} 2x + 3y &= -2 \\ -5x + 5y &= 2 \end{aligned}$

Answer

$(-16/25, -6/25)$

28) $\begin{aligned} -5x + 8y &= -3 \\ -4x - 7y &= 3 \end{aligned}$

29) $\begin{aligned} 9x + 4y &= -4 \\ -7x - 9y &= 3 \end{aligned}$

Answer

$(-24/53, 1/53)$

30) $\begin{aligned} -3x - 5y &= -4 \\ 4x + 6y &= 1 \end{aligned}$

31) $\begin{aligned} 2x + 2y &= 4 \\ 3x - 5y &= 3 \end{aligned}$

Answer

$(13/8, 3/8)$

32) $\begin{aligned} 6x - 9y &= -2 \\ -4x - 8y &= 4 \end{aligned}$

In Exercises 33-40, use the elimination method to determine how many solutions each of the following system of equations has.

33) $\begin{aligned} x + 7y &= -32 \\ -8x - 56y &= 256 \end{aligned}$

Answer

Infinite number of solutions

34) $\begin{aligned} -8x + y &= -53 \\ 56x - 7y &= 371 \end{aligned}$

35) $\begin{aligned} 16x - 16y &= -256 \\ -8x + 8y &= 128 \end{aligned}$

Answer

Infinite number of solutions

36) $\begin{aligned} 3x - 3y &= 42 \\ -6x + 6y &= -84 \end{aligned}$

37) $\begin{aligned} x - 4y &= -37 \\ 2x - 8y &= 54 \end{aligned}$

Answer

No solutions

38) $\begin{aligned} 4x + y &= -13 \\ 28x + 7y &= 189 \end{aligned}$

39) $\begin{aligned} x + 9y &= 73 \\ -4x - 5y &= -44 \end{aligned}$

Answer

One solution

40) $\begin{array}{r} 6x+y=31 \\ -5x-6y=-62 \end{array}$

4.4: Applications of Linear Systems

1) In geometry, two angles that sum to 90° are called complementary angles. If the second of two complementary angles is 42 degrees larger than 3 times the first angle, find the degree measure of both angles.

Answer

12° and 78°

2) In geometry, two angles that sum to 90° are called complementary angles. If the second of two complementary angles is 57 degrees larger than 2 times the first angle, find the degree measure of both angles.

3) The perimeter of a rectangle is 116 inches. The length of the rectangle is 28 inches more than twice the width. Find the width and length of the rectangle.

Answer

Length is 48 inches, width is 10 inches

4) The perimeter of a rectangle is 528 inches. The length of the rectangle is 24 inches more than twice the width. Find the width and length of the rectangle.

5) Maria has $\$6.35$ in change in her pocket, all in nickels and quarters. she has 59 coins in all. How many quarters does she have?

Answer

17 quarters

6) Amy has $\$5.05$ in change in her pocket, all in nickels and quarters. she has 53 coins in all. How many quarters does she have?

7) A store sells cashews for $\$6.00$ per pound and raisins for $\$7.00$ per pound. How many pounds of cashews and how many pounds of raisins should you mix to make a 50-lb mixture costing $\$6.42$ per pound?

Answer

29 pounds of cashews, 21 pounds of raisins

8) A store sells cashews for $\$3.00$ per pound and pecans for $\$8.00$ per pound. How many pounds of cashews and how many pounds of pecans should you mix to make a 50-lb mixture costing $\$4.10$ per pound?

9) Roberto has $\$5.45$ in change in his pocket, all in dimes and quarters. He has 38 coins in all. How many dimes does he have?

Answer

27 dimes

10) Benjamin has $\$7.40$ in change in his pocket, all in dimes and quarters. He has 44 coins in all. How many dimes does he have?

11) In geometry, two angles that sum to 180° are called supplementary angles. If the second of two supplementary angles is 40 degrees larger than 3 times the first angle, find the degree measure of both angles.

Answer

35° and 145°

12) In geometry, two angles that sum to 180° are called supplementary angles. If the second of two supplementary angles is 114 degrees larger than 2 times the first angle, find the degree measure of both angles.

13) Eileen inherits $\$20,000$ and decides to invest the money in two accounts, part in a certificate of deposit that pays 3% interest per year, and the rest in a mutual fund that pays 5% per year. At the end of the first year, her investments earn a total of $\$780$ in interest. Find the amount invested in each account.

Answer

$\$11,000$ in certificate of deposit, $\$9,000$ in mutual fund.

14) Alice inherits $\$40,000$ and decides to invest the money in two accounts, part in a certificate of deposit that pays 3% interest per year, and the rest in a mutual fund that pays 6% per year. At the end of the first year, her investments earn a total of $\$1,980$ in interest. Find the amount invested in each account.

15) The perimeter of a rectangle is 376 centimeters. The length of the rectangle is 12 centimeters less than three times the width. Find the width and length of the rectangle.

Answer

Length is 138 centimeters, width is 50 centimeters

16) The perimeter of a rectangle is 344 feet. The length of the rectangle is 28 feet less than three times the width. Find the width and length of the rectangle.

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CHAPTER OVERVIEW

5: Polynomial Functions

5.1: Functions
5.2: Polynomials
5.3: Applications of Polynomials
5.4: Adding and Subtracting Polynomials
5.5: Laws of Exponents
5.6: Multiplying Polynomials
5.7: Special Products
5.E: Polynomial Functions (Exercises)

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5.1: Functions

We begin with the definition of a *relation*.

Relation

A relation is a collection of ordered pairs.

The collection of ordered pairs

$$R=\{(0,3),(0,4),(1,5),(2,6)\}$$

is an example of a relation.

If we collect the first element of each ordered pair in a set, we have what is called the *domain* of the relation.

Domain

The domain of a relation is the set of all first elements of the ordered pairs.

For example, in the relation

$$R=\{(0,3),(0,4),(1,5),(2,6)\}$$

if we collect the first element of each ordered pair in R, we get the domain:

$$\text{Domain of R}=\{0,1,2\}$$

Although the number zero appears twice as a first element in the ordered pairs of R, note that we list it only once when listing the elements in the domain of R.

In similar fashion, if we collect the second elements of each ordered pair in a set, we have what is called the *range* of a relation.

Range

The range of a relation is the set of all second elements of the ordered pairs.

For example, in the relation

$$R=\{(0,3),(0,4),(1,5),(2,6)\}$$

if we collect the second element of each ordered pair in R, we get the range:

$$\text{Range of R}=\{3,4,5,6\}$$

Example $5.1.1$

State the domain and range of the relation

$$T=\{(5,3),(6,3),(7,4)\}$$

Solution

Collect the first element of each ordered pair in T to list the domain:

$$\text{Domain of } T=\{5,6,7\}$$

Collect the second element of each ordered pair in T to list the range:

$$\text{Range of } T=\{3,4\}$$

Note that even though the number three appears in the second position twice, we list it only once in describing the range.

Exercise 5.1.1

State the domain and range of the relation

$$S = \{(-1,7),(2,5),(2,3)\}$$

Answer

Domain of $S = \{-1, 2\}$, Range of $S = \{3, 5, 7\}$

Example 5.1.2

State the domain and range of the relation shown in Figure $5.1.1$.

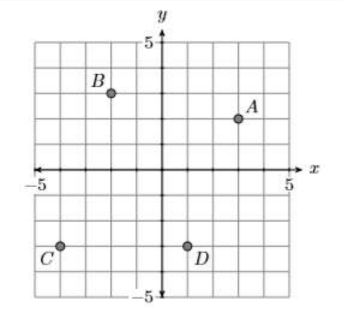


Figure $5.1.1$: We can present a relation as a collection of ordered pairs in a graph.

Solution

Point A has coordinates $(3,2)$, point B has coordinates $(-2,3)$, point C has coordinates $(-4,-3)$, and point D has coordinates $(1,-3)$. We can collect these points in a set.

$$S = \{(3,2),(-2,3),(-4,-3),(1,-3)\}$$

If we collect each element in the first position of each ordered pair, we have the domain.

$$\text{Domain of } S = \{-4,-2,1,3\}$$

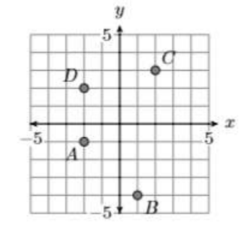
Note that it is traditional to list the domain elements in order (smallest to largest). Next, if we collect each element in the second position of each ordered pair, we have the range.

$$\text{Range of } S = \{-3,2,3\}$$

Again, it is traditional to list the elements in order. Note again that we did not repeat the number -3 in listing the range, even though it is used twice as a second element of an ordered pair in the set S.

Exercise 5.1.2

State the domain and range of the relation shown below.



Answer

Domain of $S = \{-2, 1, 2\}$, Range of $S = \{-4, -1, 2, 3\}$

Mapping Diagrams

A mapping diagram is a useful construction that helps one to analyze a relation. Consider the earlier relation $R = \{(0,3),(0,4),(1,5),(2,6)\}$, which had domain $\mathcal{D} = \{0,1,2\}$ and range $\mathcal{R} = \{3,4,5,6\}$. To construct a mapping diagram for R, list the elements in the domain of R on the left, list the elements of the range of R on the right, then use arrows to indicate the ordered pairs (see Figure $5.1.2$).

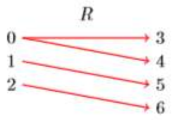


Figure $5.1.2$: Using a mapping diagram to describe the relation R.

Note how the ordred pair $(0,3)$ is indicated by drawing an arrow connecting 0 on the left to 3 on the right. We say that the relation "maps 0 to 3" and write $R: 0 \rightarrow 3$. In similar fashion:

- The ordered pair $(0,4)$ is indicated by drawing an arrow connecting 0 on the left to 4 on the right; i.e., R "maps 0 to 4" or $R: 0 \rightarrow 4$.
- The ordered pair $(1,5)$ is indicated by drawing an arrow connecting 1 on the left to 5 on the right; i.e., R "maps 1 to 5" or $R: 1 \rightarrow 5$.
- The ordered pair $(2,6)$ is indicated by drawing an arrow connecting 2 on the left to 6 on the right; i.e., R "maps 2 to 6" or $R: 2 \rightarrow 6$.

Example $5.1.3$

Create a mapping diagram for the relation in Example $5.1.1$.

Solution

The relation of Example $5.1.1$ is $T = \{(5,3),(6,3),(7,4)\}$. List the domain $\mathcal{D} = \{5,6,7\}$ on the left, the range $\mathcal{R} = \{3,4\}$ on the right, then use arrows to indicate the ordered pairs (see Figure $5.1.3$).

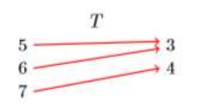


Figure $5.1.3$: A mapping diagram for the relation T.

Function Definition

A *function* is a very special type of relation.

Function

A relation is a function if and only if each object in the domain is paired with exactly one object in the range.

As a first example, consider the relation $R = \{(0,3),(0,4),(1,5),(2,6)\}$ whose mapping diagram is pictured in Figure $5.1.2$. Note that 0 in the domain is paired with two objects, 3 and 4, in the range. Hence, relation R is is not a function.

As a second example, consider the relation $T = \{(5,3),(6,3),(7,4)\}$, whose mapping diagram is pictured in Figure $5.1.3$. In this example, each domain object is paired with exactly one range object: 5 only gets sent to 3, 6 only gets sent to 3, and 7 only gets sent to 4. Hence, the relation T is a function. The fact that the range object 3 is used twice does not matter. It's the fact that each domain object gets sent to exactly one range object that matters.

Example $5.1.4$

Consider the relation pictured in Figure $5.1.4$. Is it a function?

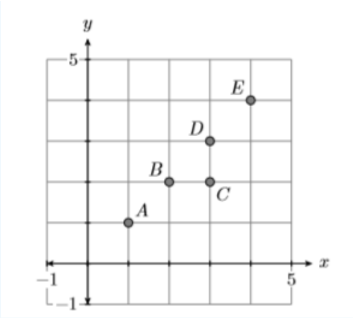


Figure 5.1.4: Is this relation a function?

Solution

The graph in Figure 5.1.4 consists of the points $A(1,1)$, $B(2,2)$, $C(3,2)$, $D(3,3)$, and $E(4,4)$. The domain is $\mathcal{D}=\{1,2,3,4\}$ and the range is $\mathcal{R}=\{1,2,3,4\}$. A mapping diagram (see Figure 5.1.5) will help us decide if the relation represented by the graph is a function. Put the domain on the left, the range on the right, then use arrows to indicate the ordered pairs. Let's call the relation f.

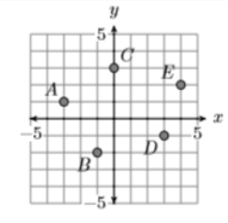
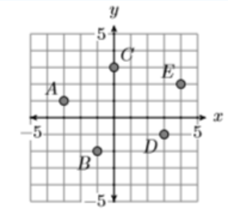


Figure 5.1.5: A mapping diagram for the relation f depicted in Figure 5.1.4.

In Figure 5.1.5, note how the domain object 3 is "sent to" or paired with two range objects, 2 and 3. Hence, the relation f is not a function.

Exercise 5.1.4

Consider the relation pictured below. Is it a function?



Answer

Yes, the relation is a function.

Example 5.1.5

The following relation pairs automobiles with their gas mileage. Determine if the relation is a function.

$$T=\{(\text{Bentley Mulsanne},18),(\text{Kia Soul},30),(\text{Lamborghini Gallardo},20),$$
$$(\text{Smart Fortwo},41),(\text{Jaguar XF},23)\}$$

Solution

In Figure 5.1.6, we create a mapping diagram indicating the relation between cars and their gas mileage. Note that each domain object on the left is paired with exactly one range object on the right. Hence, this relation is a function.

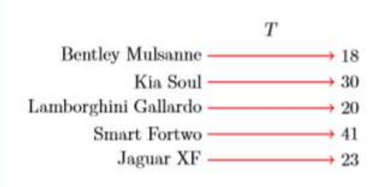

T
Bentley Mulsanne
Kia Soul
Lamborghini Gallardo
Smart Fortwo
Jaguar XF
18
30
20
41
23

Figure 5.1.6: A mapping diagram for the relation T.

Exercise 5.1.5

The following relation pairs people with their age. Determine if the relation is a function.

$$S=\{(\text{Mary},23),(\text{Joe},18),\ (\text{Alfonzo},20),(\text{Zoe},18),\ (\text{Maria},22),(\text{Chris},23)\ \}$$

Answer

Yes, the relation is a function.

Example 5.1.6

The following relation pairs a particular bird with the state that has adopted that bird as its state bird. Determine if the relation is a function.

$$R=\{\ (\text{Yellowhammer, Alabama}),\ (\text{Robin, Connecticut}),\ (\text{Nene, Hawaii}),\ (\text{Robin, Michigan})\ \}$$

Solution

In Figure 5.1.7, we create a mapping diagram indicating the relation between birds and their state adoptions. Note that the domain object “Robin” is paired with two range objects, “Connecticut” and “Michigan,” hence this relation is not a function.

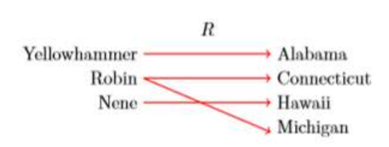

R
Yellowhammer
Robin
Nene
Alabama
Connecticut
Hawaii
Michigan

Figure 5.1.7: A mapping diagram for the relation R.

Exercise 5.1.6

The following relation pairs people with the types of cars they own. Determine if the relation is a function.

$$S=\{(\text{Bernard,station wagon}),\ (\text{Tina,truck}),\ (\text{Gilberto,sedan}),\ (\text{Kate,sport utility}),\\ (\text{Bernard,sedan}),\ (\text{Kate,minivan})\}$$

Answer

No, the relation is not a function.

Mapping Diagram Notation

The goal of this section is to introduce function notation. Let’s begin with the mapping diagram in Figure 5.1.8.

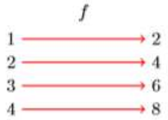


Figure 5.1.8: Mapping diagram.

The mapping diagram in Figure $5.1.8$ reveals the following facts:

- f maps 1 to 2 or $f : 1 \to 2$
- f maps 2 to 4 or $f : 2 \to 4$
- f maps 3 to 6 or $f : 3 \to 6$
- f maps 4 to 8 or $f : 4 \to 8$

Note how the notation $f : 4 \to 8$ correlates nicely with the mapping diagram in Figure $5.1.8$. The notation $f : 4 \to 8$ is read "f maps 4 to 8" or " f sends 4 to 8."

A closer look at the mapping diagram in Figure $5.1.8$ reveals an interesting pattern. The "rule" seems to be that the relation f doubles each entry in its domain: twice 1 is 2, twice 2 is 4, twice 3 is 6, etc. It's possible to give a general description of this "rule" by writing:

$$f : x \to 2x(5.1) \tag{5.1.1}$$

That is, f sends x to twice x, or equivalently, $2x$. For example, we might ask "where does f send 15?" To answer this question, we would replace x with 15 in the rule $5.1.1$ to get

$$f : 15 \to 2(15)$$

or equivalently,

$$f : 15 \to 30$$

We could also ask "where does f send -7?" To answer this question, we would replace x with -7 in the rule $5.1.1$ to get

$$f : -7 \to 2(-7)$$

or equivalently,

$$f : -7 \to -14$$

Example $5.1.7$

Given the rule $f : x \to 2x + 3$, answer the question "where does f send 8?"

Solution

To find where "f sends 8," substitute 8 for x in the rule $f : x \to 2x + 3$ to get

$$f : 8 \to 2(8) + 3$$

or equivalently,

$$f : 8 \to 19$$

Exercise $5.1.7$

Given the rule $f : x \to 3x - 5$, answer the question "where does f send -2?"

Answer

$f : -2 \to -11$

Example $5.1.8$

Given the rule $f : x \to x/(x + 3)$, answer the question "where does f send -1?"

Solution

To find where "f sends -1," substitute -1 for x in the rule $f : x \to x/(x + 3)$ to get

$$f : -1 \to \frac{-1}{-1 + 3}$$

or equivalently,

$$f:-1\rightarrow -\frac{1}{2}$$

Exercise $5.1.8$

Given the rule \(f : x → $2x^2$ +5x\), answer the question “where does f send 3?”

Answer

$f:3\rightarrow 33$

In Examples $5.1.7$ and $5.1.8$, note that each time you substitute a value for x in the given rule, you get a unique answer. This means that each object in the domain of f is sent to a unique object in the range of f, making the rules in Examples $5.1.7$ and $5.1.8$ functions. This leads us to an itemized description of a function.

Rule of Three

A function consists of three parts:

- a set of objects which mathematicians call the *domain*
- a second set of objects which mathematicians call the *range*
- a *rule* that describes how to assign each object in the domain to exactly one object in the range.

Function Notation

Although the mapping diagram notation $f:x\rightarrow 3-4x$ is quite easy to understand, the standard function notation used is $f(x)=3-4x$. With mapping diagram notation, if we want to answer the question “where does f send 12?”, we write:

$$\begin{aligned} f&:x\rightarrow 3-4x\\ f&:12\rightarrow 3-4(12)\\ f&:12\rightarrow 3-48\\ f&:12\rightarrow -45 \end{aligned}$$

Hence, $f:12\rightarrow -45$; i.e., f sends 12 to -45. Function notation uses exactly the same concept; i.e, substitute 12 for x.

Tips for Using Function Notation

1. Replace all occurrences of variables in the notation with open parentheses. Leave room between the parentheses to substitute the given value of the variable.
2. Substitute the given values of variables in the open parentheses prepared in the first step.
3. Evaluate the resulting expression according to the Rules Guiding Order of Operations.

Given $f(x)=3-4x$, to evaluate $f(12)$, first restate the function notation, then replace each occurrence of the variable with open parentheses.

$$\begin{aligned} f(x)&=3-4x && \text{Original function notation.}\\ f(\quad)&=3-4(\quad) && \text{Replace each occurrence of } x \text{ with open parentheses.} \end{aligned}$$

Now substitute 12 for x in the open parentheses prepared in the last step.

$$\begin{aligned} f(12)&=3-4(12) && \text{Substitute 12 for } x \text{ in the open parentheses positions.}\\ f(12)&=3-48 && \text{Multiply.}\\ f(12)&=-45 && \text{Subtract.} \end{aligned}$$

Hence, $f(12)=-45$; i.e., f sends 12 to -45.

Example 5.1.9

Given $f(x)=4-5x-x^2$, evaluate $f(-3)$.

Solution

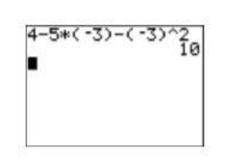
Start by replacing each occurrence of the variable x with open parentheses.

$$\begin{aligned} f(x)&=4-5x-x^2 && \text{Original function notation.}\\ f()&=4-5(\quad)-(\quad)^2 && \text{Replace each occurrence of } x \text{ with open parentheses.} \end{aligned}$$

Now substitute -3 for x in the open parentheses prepared in the last step.

$$\begin{aligned} f(-3)&=4-5(-3)-(-3)^2 && \text{Substitute } -3 \text{ for } x \text{ in the open parentheses positions.}\\ f(-3)&=4-5(-3)-9 && \text{Evaluate exponent: } (-3)^2=9\\ f(-3)&=4+15-9 && \text{Multiply: } -5(-3)=15\\ f(-3)&=10 && \text{Simplify.} \end{aligned}$$

Thus, $f(-3)=10$. Check this on your calculator (see Figure 5.1.9).



```
4-5*(-3)-(-3)^2
                10
```

Figure 5.1.9: Calculator check.

The next example demonstrates one of the advantages of function notation. For example, it is easy to refer to the function in which you want to substitute the given x-value.

Example 5.1.10

Given $f(x)=5-x$ and $g(x)=x^2-9$, find $f(-1)$ and $g(-2)$.

Solution

We're given two function definitions, f and g, but we're first asked to find $f(-1)$. This means that we must replace each occurrence of x with -1 in the function $f(x)=5-x$.

$$\begin{aligned} f(x)&=5-x && \text{Original function notation.}\\ f(\ \ )&=5-(\ \ ) && \text{Replace each occurrence of } x \text{ with open parentheses.} \end{aligned}$$

Now substitute -1 for x in the open parentheses prepared in the last step.

$$\begin{aligned} f(-1)&=5-(-1) && \text{Substitute } -1 \text{ for } x \text{ in the open}\\ &=5+1 && \text{Add the opposite.}\\ &=6 && \text{Simplify.} \end{aligned}$$

Thus, $f(-1)=6$. We're next asked to find $g(-2)$. This means that we must replace each occurrence of x with -2 in the function $g(x)=x^2-9$.

$$\begin{aligned} g(x)&=x^2-9 && \text{Original function notation.}\\ g(\ \ )&=(\ \ )^2-9 && \text{Replace each occurrence of } x \text{ with open parentheses.} \end{aligned}$$

Now substitute -2 for x in the open parentheses prepared in the last step.

$$\begin{aligned} g(-2)&=(-2)^2-9 && \text{Substitute } -2 \text{ for } x \text{ in the open parentheses position.}\\ &=4-9 && \text{Exponent first: } (-2)^2=4\\ &=-5 && \text{Simplify.} \end{aligned}$$

Thus, $g(-2)=-5$.

Exercise 5.1.10

Given f\((x)=3x^2 −20\) and \(g(x)=4x +6/x\), find $f(-3)$ and $g(2)$.

Answer

$f(-3)=7$ and $g(2)=10$

Interchanging y and $f(x)$

In most cases, y and $f(x)$ are completely interchangeable. For example, compare and contrast the following two examples.

Question: Given $y=3x+7$, find y when x equals 5.

Solution: Replace x with 5.

$$\begin{aligned}&y=3x+7\\&y=3(5)+7\\&y=15+7\\&y=22\end{aligned}$$

Question: Given $f(x)=3x+7$, evaluate $f(5)$

Solution: Replace x with 5.

$$\begin{aligned}&f(x)=3x+7\\&f(5)=3(5)+7\\&f(5)=15+7\\&f(5)=22\end{aligned}$$

In each case, the answer is 22. However, in the first case, the answer $y=22$ disguises the fact that an x-value of 5 was used to arrive at the result. On the other hand, when we use function notation, the final answer $f(5)=22$ indicates that we used an x-value of 5 to determine that the y-value is 22. This is another advantage of function notation.

Let's look at one final application demonstrating that y and $f(x)$ are interchangeable.

Example 5.1.11

Sketch the graph of $f(x)=2x-3$.

Solution

Because y and $f(x)$ are interchangeable, the instruction is identical to "sketch the graph of $y=2x-3$." The graph is a line, with slope 2 and y -intercept at $(0,-3)$. Plot the y-intercept at $(0,-3)$, then move up 2 and right 1 to create a line with slope 2 (see Figure $5.1.10$). Note how we've labeled the graph with its equation using function notation.

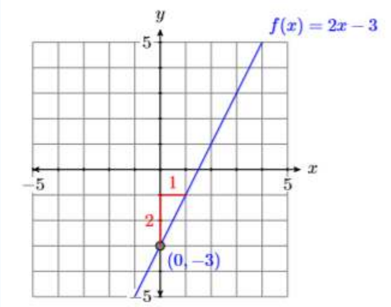
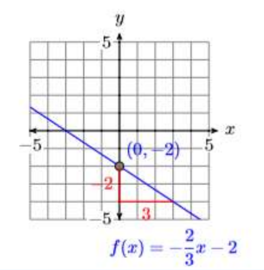


Figure $5.1.10$: The graph of $f(x)=2x-3$ is a line.

Exercise 5.1.11

Sketch the graph of $f(x) = -\frac{2}{3}x - 2$

Answer



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5.2: Polynomials

We begin with the definition of a term.

Definition: Term

A *term* is either a single number (called a *constant* term) or the product of a number and one or more variables.

For example, each of the following is a term.

$$-5 \quad -3x^2 \quad 12y^2z^3 \quad 13a^2bc^3$$

Note how the first term is a single number, while the remaining terms are products of a number and one or more variables. For example, $-3x^2$ is the product of -3, x, and x.

Definition: Coefficient

When a term is a product of a number and one or more variables, the number is called the *coefficient* of the term. In the case of a term that is a single number, the number itself is called the coefficient.

Thus, for example, the coefficients of the terms

$$-5 \quad -3x^2 \quad 12y^2z^3 \quad 13a^2bc^3$$

are -5, -3, 12, and 13, respectively.

Definition: Degree

The *degree* of a term is the sum of the exponents on each variable of the term. A constant term (single number with no variables) has degree zero.

Thus, for example, the degrees of the terms

$$-5 \quad -3x^2 \quad 12y^2z^3 \quad 13a^2bc^3$$

are 0, 2, 5, and 6, respectively. In the last example, note that $13a^2bc^3$ is equivalent to $13a^2b^1c^3$, so adding exponents, we get:

$$\begin{aligned} \text{Degree of } 13a^2bc^3 &= \text{ Degree of } 13a^2b^1c^3 \\ &= 2+1+3 \\ &= 6 \end{aligned}$$

Definition: Monomial

The words *monomial* and *term* are equivalent.

Thus,

$$-5 \quad -3x^2 \quad 12y^2z^3 \quad 13a^2bc^3$$

are monomials.

Definition: Binomial

A *binomial* is a mathematical expression containing exactly two terms, separated by plus or minus signs.

For example, each of the mathematical expressions

$$2x+3y \quad -3a^2-3b^2 \quad xy+7 \quad -3x^2y+5xy^2$$

is a binomial. Each expression has exactly two terms.

Definition: Trinomial

A *trinomial* is a mathematical expression containing exactly three terms, separated by plus or minus signs.

For example, each of the mathematical expressions

$$2x^2+3x+7 \quad a^2+2ab+b^2 \quad x^4-2x^2y^2+3y^4$$

is a trinomial. Each expression has exactly three terms.

A bicycle has two wheels, a binomial has two terms. A tricycle has three wheels, a trinomial has three terms. But once we get past three terms, the assignment of special names ceases and we use the generic word *polynomial*, which means "many terms."

Definition: Polynomial

A *polynomial* is a many-termed mathematical expression, with terms separated by plus or minus signs. The coefficients of a polynomial are the coefficients of its terms.

Each of the previous expressions,

$$12y^2z^3 \quad -3a^2-3b^2 \quad x^4-2x^2y^2+3y^4$$

though assigned the particular names monomial, binomial, and trinomial, respectively, are also "many-termed" expressions and can also be called polynomials. However, because the word polynomial means "many terms," we can also use the word polynomial to describe mathematical expressions with more than three terms, such as:

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4$$

The coefficients of $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ are 1, -4, 6, -4, and 1.

Ascending and Descending Powers

When asked to simplify a polynomial expression, we should combine any like terms we find, and when possible, arrange the answer in *ascending* or *descending* powers.

Example $5.2.1$

Simplify the following polynomial expression, arranging your answer in the descending powers of x. Once you've completed that task, make a second arrangement, ordering your terms in ascending powers of x.

$$2x^3+7x-3x^2+11x+8x^2+11+15x$$

Solution

In order to arrange our answer in descending powers of x, we want to place the term with the highest power of x first and the term with the lowest power of x last. We use the commutative and associative properties to change the order and regroup, then we combine like terms.

$$\begin{aligned} &2x^3+7x-3x^2+11x+8x^2+11+15x \\ &\quad=2x^3+\left(-3x^2+8x^2\right)+(7x+11x+15x)+11 \\ &\quad=2x^3+5x^2+33x+11 \end{aligned}$$

Note how the powers of x start at 3, then go down in order.

To arrange our final answer in ascending powers of x, we put the lowest power of x first, then the highest power of x last, regrouping and combining like terms.

$$\begin{aligned} &2x^3+7x-3x^2+11x+8x^2+11+15x \\ &\quad=11+(7x+11x+15x)+\left(-3x^2+8x^2\right)+2x^3 \\ &\quad=11+33x+5x^2+2x^3 \end{aligned}$$

Note how we start with the constant term, then the powers of x increase in order.

Exercise 5.2.1

Simplify the following polynomial, and arrange your answer in ascending powers of

$$3x^2-5x^3+8x+9x^2-7x+2x^3$$

Answer

$x+12x^2-3x^3$

When we have a polynomial in a single variable, such as the polynomial in Example 5.2.1, arranging the terms in ascending or descending order is fairly straightforward. However, a polynomial in two or more variables is a bit more difficult, and sometimes impossible, to arrange in a decent order.

Example 5.2.2

Simplify the following polynomial expression, then arrange your answer in descending powers of x.

$$x^3+2xy^2-6x^2y+y^3-3xy^2+4x^2y$$

Solution

We'll again use the commutative and associative properties to change the order and regroup, putting the terms with the highest powers of x first, then follow with terms containing lower powers of x in order.

$$\begin{aligned} x^3+2xy^2-6x^2y+y^3-3xy^2+4x^2y \\ &= x^3+\left(-6x^2y+4x^2y\right)+\left(2xy^2-3xy^2\right)+y^3 \\ &= x^3-2x^2y-xy^2+y^3 \end{aligned}$$

Note that this is a very natural order, the powers of x decrease while simultaneously the powers of y increase.

Exercise 5.2.2

Simplify the following polynomial, and arrange your answer in descending powers of x:

$$-4x^2y^2+3xy^3+6x^3y-xy^3+2x^2y^2$$

Answer

$6x^3y-2x^2y^2+2xy^3$

Not all examples will have nice ordering presented in Example 5.2.2, with the powers of one variable descending while the powers of the other variable simultaneously ascends. Sometimes we have to make some very subjective choices on the ordering of terms.

Example 5.2.3

Simplify the following polynomial expression, then arrange your answer in some sort of reasonable order.

$$a^3b^3+2a^2b-3a^2b^3+4a^3b^3+5a^4+3a^2b+b^5$$

Solution

Let's try to arrange the terms so that the powers of a descend. Again, we use the commutative and associative properties to change the order and regroup.

$$\begin{aligned} a^3b^3+2a^2b-3a^2b^3+4a^3b^3+5a^4+3a^2b+b^5 \\ &= 5a^4+\left(a^3b^3+4a^3b^3\right)+\left(2a^2b+3a^2b\right)-3a^2b^3+b^5 \\ &= 5a^4+5a^3b^3+5a^2b-3a^2b^3+b^5 \end{aligned}$$

Note that in our final arrangement, the powers of a descend, but the powers of b bounce up and down, but at least we have the powers of a descending. That should help us spot if we've missed a term while simplifying the given problem.

Exercise 5.2.3

Simplify the following polynomial, and arrange your answer in ascending powers of b:

$$5a^3b^2 + 4ab^3 - 2a^2b + 3a^3b^2 - ab^3$$

Answer

$-2a^2b + 8a^3b^2 + 3ab^3$

The Degree of a Polynomial

To find the degree of a polynomial, locate the term of the polynomial having the highest degree.

The degree of a polynomial

The *degree* of a polynomial is the degree of the term having the highest degree.

Finding the degree of a polynomial of a single variable is pretty easy.

Example 5.2.4

What is the degree of the polynomial $x^3 - 4x^2 + 5 - 6x + 2x^7$?

Solution

First, let’s arrange the polynomial in descending powers of x.

$$2x^7 + x^3 - 4x^2 - 6x + 5$$

Arranging the polynomial in descending powers of x makes it easier to see that the term of the polynomial with the highest degree is $2x^7$. Therefore, the degree of the polynomial is 7.

Exercise 5.2.4

What is the degree of the polynomial $2x^3 + 8x^2 + 3x^4 + 2x + 10$?

Answer

4

Finding the degree of a polynomial of more than one variable is a little bit trickier.

Example 5.2.5

What is the degree of the polynomial $x^4 - 2x^3y^7 + y^5$?

Solution

Note that the polynomial is already arranged in descending powers of x, an arrangement that is probably as good as we are going to get. In the following table, we list the degree of each term. Remember, the degree of any term is found by summing the exponents on its variables.

Term	Degree
x^4	4
$-2x^3y^7$	10
y^5	5

Hence, the term with the highest degree is $-2x^3y^7$, making 10 the degree of the polynomial.

Exercise 5.2.5

What is the degree of the polynomial $x^2y^4 - 6x^2y^2 + 5x^2y^5 - 2xy$?

Answer

7

Polynomial Functions

First we define what we mean by a polynomial function.

Polynomial function

A polynomial function is a function defined by a rule that assigns to each domain object a range object defined by a polynomial expression.

Advanced courses, such as multivariate calculus, frequently use polynomial functions of more than one variable such as $f(x, y) = x^2 + y^2$. However, in this course, our focus will be on polynomial functions of a single variable, such as $p(x) = 3 - 4x - 9x^2$ and $q(x) = x^3 - 9x^2 + 11$.

Example 5.2.6

Given the polynomial function $p(x) = x^3 - 8x - 11$, evaluate $p(-3)$.

Solution

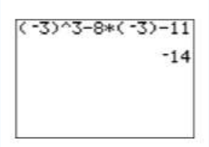
To evaluate $p(-3)$, first restate the function definition, then replace each occurrence of the variable x with open parentheses.

$$\begin{aligned}
p(x) &= x^3 - 8x - 11 && \text{Original function definition.} \\
p(\quad) &= (\quad)^3 - 8(\quad) - 11 && \text{Replace each occurrence of } x \text{ with open parentheses.}
\end{aligned}$$

Next, substitute -3 for x in the open parentheses prepared in the last step.

$$\begin{aligned}
p(-3) &= (-3)^3 - 8(-3) - 11 && \text{Substitute } -3 \text{ for } x \text{ in the open parentheses positions.} \\
p(-3) &= -27 - 8(-3) - 11 && \text{Exponent first: } (-3)^3 = -27 \\
p(-3) &= -27 + 24 - 11 && \text{Multiply: } -8(-3) = 24 \\
p(-3) &= -14 && \text{Add.}
\end{aligned}$$

Hence, $p(-3) = -14$. You can easily check this result on your calculator (see Figure 5.2.1).



```
(-3)^3-8*(-3)-11
             -14
```

Figure 5.2.1: Calculator check.

Exercise 5.2.6

Given the polynomial function $p(x) = -3x^2 + 7x + 4$, evaluate $p(2)$.

Answer

6

The Graph of a Polynomial Function

One of the most important polynomial functions in all of mathematics and science is the polynomial having degree two.

Quadratic polynomial

The second degree polynomial having the form

$$p(x)=ax^2+bx+c$$

is called a *quadratic polynomial*. The graph of this polynomial is called a *parabola*.

The parabola is approximately U-shaped. Some open upwards, some open downwards, depending on the sign of the leading term.

In Figure $5.2.2$, the leading term of the parabola $p(x)=2x^2-8x+6$ has positive two as its coefficient, so it opens upward.

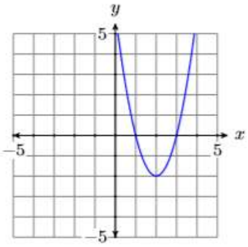


Figure $5.2.2$: The graph of $p(x)=2x^2-8x+6$ opens up.

In Figure $5.2.3$, the leading term of the parabola $p(x)=-2x^2-8x-6$ has negative two as its coefficient, so it opens downward.

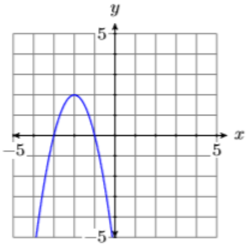


Figure $5.2.3$: The graph of $p(x)=-2x^2-8x-6$ opens down.

Note

The sign of the leading term of $p(x)=ax^2+bx+c$ determines whether the parabola opens up or down.

- If $a>0$, the parabola opens upward.
- If $a<0$, the parabola opens downward.

The turning point of a parabola has a special name.

The vertex of a parabola

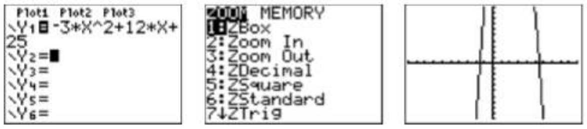
The graph of the second degree polynomial $p(x)=ax^2+bx+c$ has a single turning point, called the *vertex* of the parabola.

Example $5.2.7$

Use your graphing calculator to sketch the graph of the quadratic polynomial $p(x)=-3x^2+12x+25$.

Solution

The degree of the polynomial $p(x)=-3x^2+12x+25$ is two, so it is a quadratic polynomial and its graph is a parabola. Moreover, its leading term has negative three as its coefficient, so we know that the parabola opens downward. Enter $y=-3x^2+12x+25$ as $Y1=-3*X\wedge 2+12*X+25$ in the **Y= menu** (see the first image in Figure $5.2.4$), then select **6:ZStandard** from the **ZOOM** menu to produce the third image in Figure $5.2.4$.

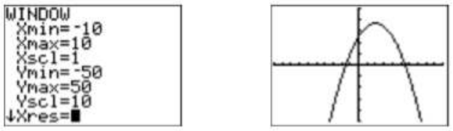


```
Plot1 Plot2 Plot3
\Y1■-3*X^2+12*X+
25
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

```
ZOOM MEMORY
1:ZBox
2:Zoom In
3:Zoom Out
4:ZDecimal
5:ZSquare
6:ZStandard
7↓ZTrig
```

Figure 5.2.4: Sketching the graph of $p(x) = -3x^2 + 12x + 25$.

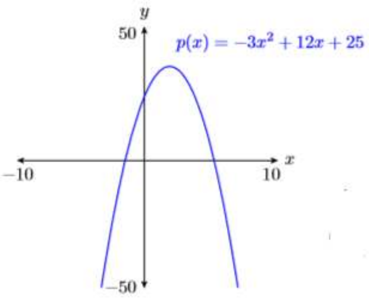
Note that the graph in Figure 5.2.4 appears to have the U-shape of a parabola that opens downwards. Its vertex (turning point) is not visible, but one would surmise that it lies off the top of the screen. We need to adjust the **WINDOW** parameters so that the vertex of the parabola is visible in the viewing screen. After some experimentation, we settle on the parameters shown in the first image in Figure 5.2.5, then push the **GRAPH** button to produce the second image in Figure 5.2.5.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-50
 Ymax=50
 Yscl=10
↓Xres=■
```

Figure 5.2.5: Adjust the **WINDOW** parameters so that the vertex is visible in the viewing screen.

In reporting your result on your homework, follow the *Calculator Submission Guidelines* from Chapter 3, Section2.

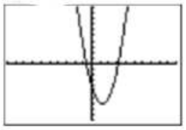


1. Draw axes with a ruler.
2. Label the horizontal axis x and the vertical axis y.
3. Indicate the **WINDOW** parameters $\mathrm{Xmin}, \mathrm{Xmax}, \mathrm{Ymin}$, and Ymax\) at the end of each axis.
4. Freehand the curve and label it with its equation.

Exercise 5.2.7

Use your graphing calculator to sketch the graph of the quadratic polynomial $p(x) = 2x^2 - 5x - 4$.

Answer



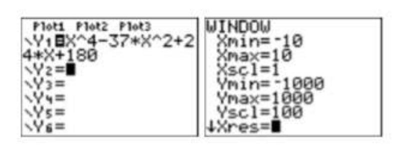
When the degree of the polynomial is larger than two, the number of turning points of the graph might increase. This makes for some very interesting curves. In more advanced courses, such as intermediate and college algebra, you will be introduced to a variety of techniques that will help you determine appropriate viewing windows for the graphs of these higher degree polynomials. However, in this introductory section, we will assist you by suggesting a good viewing window for each polynomial, one that will allow you to see all of the turning points of the graph of the polynomial.

Example $5.2.8$

Use your graphing calculator to sketch the graph of the polynomial function $p(x) = x^4 - 37x^2 + 24x + 180$. Set your window parameters as follows: $\mathbf{X}\min = -10$, $\mathbf{X}\max = 10$, $\mathbf{X}\mathrm{scl} = 1$, $\mathbf{Y}\min = -1000$, $\mathbf{Y}\max = 1000$, and $\mathbf{Y}\mathrm{scl} = 100$.

Solution

Enter the polynomial function in $\mathbf{Y1}$ of the **Y= menu**, then enter the suggested window parameters in the **WINDOW** menu (see Figure $5.2.6$).



```
Plot1 Plot2 Plot3
\Y1■X^4-37*X^2+2
4*X+180
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-1000
 Ymax=1000
 Yscl=100
↓Xres=■
```

Figure $5.2.6$: Enter the polynomial and adjust the WINDOW parameters.

Push the **GRAPH** button on the top row of your calculator to produce the graph of the polynomial function shown in Figure $5.2.7$.

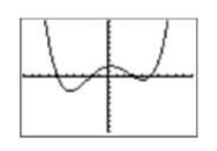


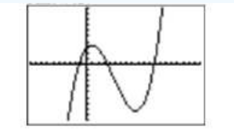
Figure $5.2.7$: The graph of $p(x) = x^4 - 37x^2 + 24x + 180$.

Sweet-looking curve!

Exercise $5.2.8$

Use your graphing calculator to sketch the graph of the quadratic polynomial p(x)=x3 −14x2 + $20x$+ 60. Set your window parameters as follows: $\mathbf{X}\min = -10$, $\mathbf{X}\max = 20$, $\mathbf{X}\mathrm{scl} = 1$, $\mathbf{Y}\min = -200$, $\mathbf{Y}\max = 200$, and $\mathbf{Y}\mathrm{scl} = 20$.

Answer



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5.3: Applications of Polynomials

In this section we investigate real-world applications of polynomial functions.

Example 5.3.1

The average price of a gallon of gas at the beginning of each month for the period starting in November 2010 and ending in May 2011 are given in the margin. The data is plotted in Figure 5.3.1 and fitted with the following third degree polynomial, where t is the number of months that have passed since October of 2010.

$$p(t) = -0.0080556t^3 + 0.11881t^2 - 0.30671t + 3.36 \tag{5.3.1}$$

Use the graph and then the polynomial to estimate the price of a gallon of gas in California in February 2011.

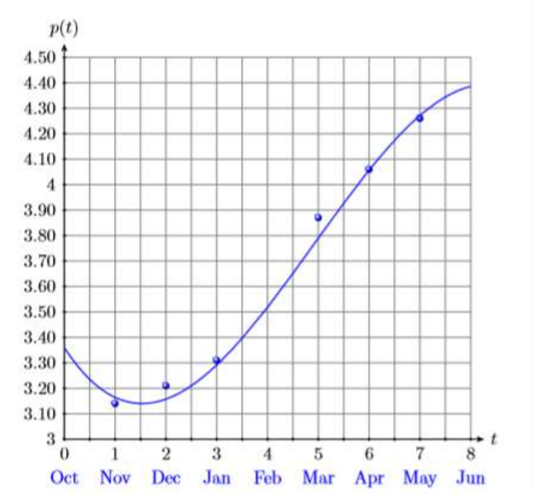


Figure 5.3.1: Fitting gas price versus month with a cubic polynomial.

Month	Price
Nov.	3.14
Dec.	3.21
Jan	3.31
Mar.	3.87
Apr.	4.06
May	4.26

Solution

Locate February ($t = 4$) on the horizontal axis. From there, draw a vertical arrow up to the graph, and from that point of intersection, a second horizontal arrow over to the vertical axis (see Figure 5.3.2). It would appear that the price per gallon in February was approximately $\$3.51$.

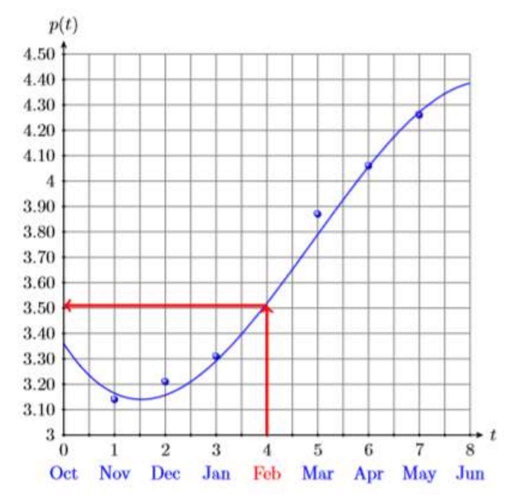
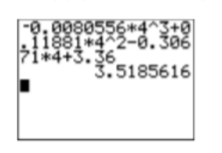


Figure $5.3.2$: Approximating price of gas during February.

Next, we'll use the fitted third degree polynomial to approximate the price per gallon for the month of February, 2011. Start with the function defined by Equation $5.3.1$ and substitute 4 for t.

$$\begin{aligned} p(t) &= -0.0080556t^3 + 0.11881t^2 - 0.30671t + 3.36 \\ p(4) &= -0.0080556(4)^3 + 0.11881(4)^2 - 0.30671(4) + 3.36 \end{aligned}$$

Use the calculator to evaluate $p(4)$ (see Figure $5.3.3$). Rounding to the nearest penny, the price in February was $\$3.52$ per gallon.



```
-0.0080556*4^3+0
.11881*4^2-0.306
71*4+3.36
       3.5185616
█
```

Figure $5.3.3$: Evaluating $p(4)$.

Example $5.3.2$

If a projectile is fired into the air, its height above ground at any time is given by the formula

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{5.3.2}$$

where

y = height above ground at time t

y_0 = initial height above ground at time $t = 0$

v_0 = initial velocity at time $t = 0$

g = acceleration due to gravity

t = time passed since projectile's firing

If a projectile is launched with an initial velocity of 100 meters per second (100m/s) from a rooftop 8 meters (8m) above ground level, at what time will the projectile first reach a height of 400 meters (400m)? Note: Near the earth's surface, the acceleration due to gravity is approximately 9.8 meters per second per second (9.8(m/s)/s or 9.8m/s^2).

Solution

We're given the initial height is $y_0 = 8$m, the initial velocity is $v_0 = 100$m/s, and the acceleration due to gravity is $g = 9.8$m/s^2. Substitute these values in Equation $5.3.2$, then simplify to produce the following result:

$$y = y_0 + v_0 t - \frac{1}{2} g t^2$$

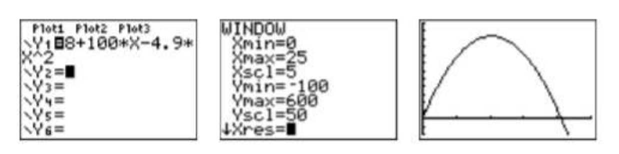
$$y = y_0 + v_0t - \frac{}{2}gt^2$$

$$y = 8 + 100t - \frac{1}{2}(9.8)t^2$$

$$y = 8 + 100t - 4.9t^2$$

Enter $y = 8 + 100t - 4.9t^2$ as **Y1** $=$ **8** $+$ 100***X** $-$ 4.9***X** $\wedge$ **2** in the **Y= menu** (see the first image in Figure 5.3.4). After some experimentation, we settled on the **WINDOW** parameters shown in the second image in Figure 5.3.4. Push the GRAPH button to produce the graph of $y = 8 + 100t - 4.9t^2$ shown in the third image Figure 5.3.4.

In this example, the horizontal axis is actually the t-axis. So when we set Xmin and Xmax, we're actually setting bounds on the t-axis.



```
Plot1 Plot2 Plot3
\Y1=8+100*X-4.9*
X^2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

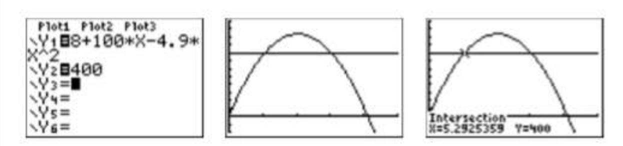
```
WINDOW
 Xmin=0
 Xmax=25
 Xscl=5
 Ymin=-100
 Ymax=600
 Yscl=50
↓Xres=
```

Figure 5.3.4: Sketching the graph of $y = 8 + 100t - 4.9t^2$.

To find when the projectile reaches a height of 400 meters (400m), substitute 400 for y to obtain:

$$400 = 8 + 100t - 4.9t^2 \tag{5.3.3}$$

Enter the left-hand side of Equation 5.3.3 into **Y2** in the **Y= menu**, as shown in the first image in Figure 5.3.5. Push the **GRAPH** button to produce the result shown in the second image in Figure 5.3.5. Note that there are two points of intersection, which makes sense as the projectile hits 400 meters on the way up and 400 meters on the way down.



```
Plot1 Plot2 Plot3
\Y1=8+100*X-4.9*
X^2
\Y2=400
\Y3=
\Y4=
\Y5=
\Y6=
```

```
Intersection
X=5.2925359  Y=400
```

Figure 5.3.5: Determining when the object first reaches 400 meters.

To find the first point of intersection, select **5:intersect** from the **CALC** menu. Press **ENTER** in response to "First curve," then press ENTER again in response to "Second curve." For your guess, use the arrow keys to move the cursor closer to the first point of intersection than the second. At this point, press **ENTER** in response to "Guess." The result is shown in the third image in Figure 5.3.5. The projectile first reaches a height of 400 meters at approximately 5.2925359 seconds after launch.

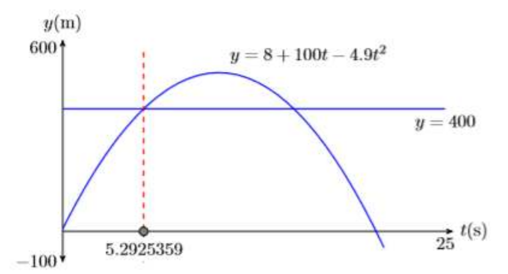


Figure 5.3.6: Reporting your graphical solution on your homework.

The parabola shown in Figure 5.3.6 is not the actual flight path of the projectile. The graph only predicts the height of the projectile as a function of time.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with t and y, respectively (see Figure 5.3.6). Include the units (seconds (s) and meters (m)).
- Place your **WINDOW** parameters at the end of each axis (see Figure 5.3.6). Include the units (seconds (s) and meters (m)).
- Label each graph with its equation (see Figure 5.3.6).

- Draw a dashed vertical line through the first point of intersection. Shade and label the point (with its t-value) where the dashed vertical line crosses the t-axis. This is the first solution of the equation $400 = 8 + 100t - 4.9t^2$ (see Figure $5.3.6$).

Rounding to the nearest tenth of a second, it takes the projectile approximately $t \approx 5.3$ seconds to first reach a height of 400 meters.

The phrase "shade and label the point" means fill in the point on the t-axis, then write the t-value of the point just below the shaded point.

Exercise $5.3.2$

If a projectile is launched with an initial velocity of 60 meters per second from a rooftop 12 meters above ground level, at what time will the projectile first reach a height of 150 meters?

Answer

≈ 3.0693987 seconds

Zeros and x-intercepts of a Function

Recall that $f(x)$ and y are interchangeable. Therefore, if we are asked to find where a function is equal to zero, then we need to find the points on the graph of the function that have a y-value equal to zero (see Figure $5.3.7$).

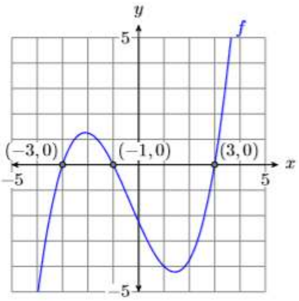


Figure $5.3.7$: Locating the zeros of a function.

Zeros and x-intercepts

The points where the graph off crosses the x-axis are called the x-intercepts of the graph of f. The x-value of each x-intercept is called a zero of the function f.

The graph of f crosses the x-axis in Figure $5.3.7$ at $(-3, 0)$, $(-1, 0)$, and $(3, 0)$. Therefore:

- The x-intercepts of f are: $(-3, 0)$, $(-1, 0)$, and $(3, 0)$
- The zeros of f are: -3, -1, and 3

Key idea

A function is zero where its graph crosses the x-axis.

Example $5.3.3$

Find the zero(s) of the function $f(x) = 1.5x + 5.25$.

Solution

Remember, $f(x) = 1.5x + 5.25$ and $y = 1.5x + 5.25$ are equivalent. We're looking for the value of x that makes $y = 0$ or $f(x) = 0$. So, we'll start with $f(x) = 0$, then replace $f(x)$ with $1.5x + 5.25$.

$$\begin{aligned} f(x) &= 0 && \text{We want the value of } x \text{ that makes the function equal to zero.} \\ 1.5x + 5.25 &= 0 && \text{Replace } f(x) \text{ with } 1.5x + 5.25 \end{aligned}$$

Now we solve for x.

$$\begin{aligned} 1.5x &= -5.25 & \color{red}{\text{Subtract } 5.25 \text{ from both sides.}} \\ x &= \frac{-5.25}{1.5} & \color{red}{\text{Divide both sides by } 1.5} \\ x &= -3.5 & \color{red}{\text{Divide: } -5.25/1.5 = -3.5} \end{aligned}$$

Check: Substitute -3.5 for x in the function $f(x) = 1.5x + 5.25$.

$$\begin{aligned} f(x) &= 1.5x + 5.25 & \color{red}{\text{The original function.}} \\ f(-3.5) &= 1.5(-3.5) + 5.25 & \color{red}{\text{Substitute } -3.5 \text{ for } x} \\ f(-3.5) &= -5.25 + 5.25 & \color{red}{\text{Multiply: } 1.5(-3.5) = -5.25} \\ f(-3.5) &= 0 & \color{red}{\text{Add.}} \end{aligned}$$

Note that -3.5, when substituted for x, makes the function $f(x) = 1.5x + 5.25$ equal to zero. This is why -3.5 is called a zero of the function.

Graphing calculator solution: We should be able to find the zero by sketching the graph of f and noting where it crosses the x-axis. Start by loading the function $f(x) = 1.5x + 5.25$ into **Y1** in the **Y= menu** (see the first image in Figure $5.3.8$).

Select **6:ZStandard** from the **ZOOM** menu to produce the graph of f (see the second image in Figure $5.3.8$). Press **2ND CALC** to open the the **CALCULATE** menu (see the third image in Figure $5.3.8$). To find the zero of the function f:

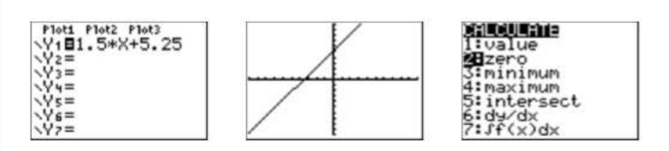


Figure $5.3.8$: Finding the zero of $f(x) = 1.5x + 5.25$.

1. Select **2:zero** from the **CALCULATE** menu. The calculator responds by asking for a "Left Bound?" (see the first image in Figure $5.3.9$). Use the left arrow button to move the cursor so that it lies to the left of the x-intercept of f and press **ENTER**.
2. The calculator responds by asking for a "Right Bound?" (see the second image in Figure $5.3.9$). Use the right arrow button to move the cursor so that it lies to the right of the x-intercept of f and press **ENTER**.
3. The calculator responds by asking for a "Guess?" (see the third image in Figure $5.3.9$). As long as your cursor lies between the left- and right-bound marks at the top of the screen (see the third image in Figure $5.3.9$), you have a valid guess. Since the cursor already lies between the left- and right-bounds, simply press **ENTER** to use the current position of the cursor as your guess.

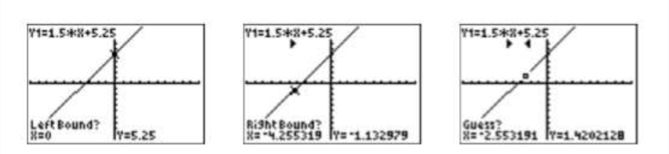


Figure $5.3.9$: Using **2:zero** from the **CALCULATE** menu.

The calculator responds by approximating the zero of the function as shown in Figure $5.3.10$.

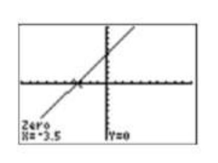


Figure $5.3.10$: -3.5 is a zero off.

Note that the approximation found using the calculator agrees nicely with the zero found using the algebraic technique.

Exercise 5.3.3

Find the zero(s) of the function $f(x)=2.6x-9.62$.

Answer

3.7

Example 5.3.4

How long will it take the projectile in Example 5.3.2 to return to ground level?

Solution

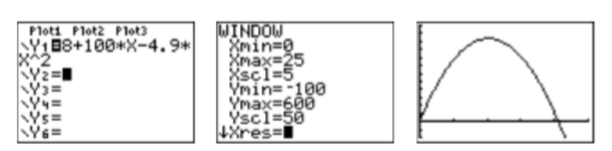
In Example 5.3.2, the height of the projectile above the ground as a function of time is given by the equation

$$y=8+100t-4.9t^2$$

When the projectile returns to the ground, its height above ground will be zero meters. To find the time that this happens, substitute $y=0$ in the last equation and solve for t.

$$0=8+100t-4.9t^2$$

Enter the equation $y=8+100t-4.9t^2$ into **Y1** in the **Y= menu** of your calculator (see the first image in Figure 5.3.11), then set the **WINDOW** parameters shown in the second image in Figure 5.3.11. Push the **GRAPH** button to produce the graph of the function shown in the third image in Figure 5.3.11.



```
Plot1 Plot2 Plot3
\Y1■8+100*X-4.9*
X^2
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=0
 Xmax=25
 Xscl=5
 Ymin=-100
 Ymax=600
 Yscl=50
↓Xres=■
```

Figure 5.3.11: Sketching the graph of $y=8+100t-4.9t^2$.

In this example, the horizontal axis is actually the t-axis. So when we set Xmin and Xmax, we're actually setting bounds on the t-axis.

To find the time when the projectile returns to ground level, we need to find where the graph of $y=8+100t-4.9t^2$ crosses the horizontal axis (in this case the t-axis) . Select **2:zero** from the **CALC** menu. Use the arrow keys to move the cursor slightly to the left of the t-intercept, then press **ENTER** in response to "Left bound." Move your cursor slightly to the right of the t-intercept, then press **ENTER** in response to "Right bound." Leave your cursor where it is and press **ENTER** in response to "Guess." The result is shown in Figure 5.3.11.

- Label the horizontal and vertical axes with t and y, respectively (see Figure 5.3.12). Include the units (seconds (s) and meters (m)).
- Place your **WINDOW** parameters at the end of each axis (see Figure 5.3.12).
- Label the graph with its equation (see Figure 5.3.12).
- Draw a dashed vertical line through the t-intercept. Shade and label the t-value of the point where the dashed vertical line crosses the t-axis. This is the solution of the equation $0=8+100t-4.9t^2$ (see Figure 5.3.12).

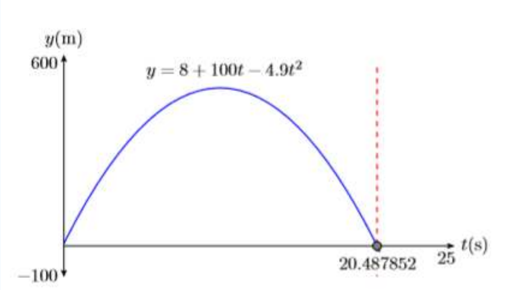


Figure 5.3.12: Reporting your graphical solution on your homework.

Rounding to the nearest tenth of a second, it takes the projectile approximately $t \approx 20.5$ seconds to hit the ground.

Exercise $5.3.4$

If a projectile is launched with an initial velocity of 60 meters per second from a rooftop 12 meters above ground level, at what time will the projectile return to ground level?

Answer

≈ 12.441734 seconds

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5.4: Adding and Subtracting Polynomials

In this section we concentrate on adding and subtracting polynomial expressions, based on earlier work combining like terms in Ascending and Descending Powers. Let's begin with an addition example.

Example $5.4.1$

Simplify:

$$\left(a^2+3ab-b^2\right)+\left(4a^2+11ab-9b^2\right)$$

Solution

Use the commutative and associative properties to change the order and regroup. Then combine like terms.

$$\begin{aligned} &\left(a^2+3ab-b^2\right)+\left(4a^2+11ab-9b^2\right)\\ &\quad=\left(a^2+4a^2\right)+(3ab+11ab)+\left(-b^2-9b^2\right)\\ &\quad=5a^2+14ab-10b^2 \end{aligned}$$

Exercise $5.4.1$

Simplify: $(3s^2-2st+4t^2)+(s^2+7st-5t^2)$

Answer

$4s^2+5st-t^2$

Let's combine some polynomial functions.

Example $5.4.2$

Given $f(x)=3x^2-4x-8$ and $g(x)=x^2-11x+15$, simplify $f(x)+g(x)$.

Solution

First, replace $f(x)$ and $g(x)$ with their definitions. Be sure to surround each polynomial with parentheses, because we are asked to add all of $f(x)$ to all of $g(x)$.

$$f(x)+g(x)=(3x^2-4x-8)+(x^2-11x+15)$$

Now use the commutative and associative properties to change the order and regroup. Combine like terms.

$$\begin{aligned} &=(3x^2+x^2)+(-4x-11x)+(-8+15)\\ &=4x^2-15x+7 \end{aligned}$$

Hence, $f(x)+g(x)=4x^2-15x+7$.

Exercise $5.4.2$

Given $f(x)=2x^2+9x-5$ and $g(x)=-x^2-4x+3$, simplify $f(x)+g(x)$.

Answer

x^2+5x-2

If you are comfortable skipping a step or two, it is not necessary to write down all of the steps shown in Examples $5.4.1$ and $5.4.2$. Let's try combining like terms mentally in the next example.

Example 5.4.3

Simplify:

$$\left(x^{3}-2x^{2}y+3xy^{2}+y^{3}\right)+\left(2x^{3}-4x^{2}y-8xy^{2}+5y^{3}\right)$$

Solution

If we use the associative and commutative property to reorder and regroup, then combine like terms, we get the following result.

$$\begin{aligned}&\left(x^{3}-2x^{2}y+3xy^{2}+y^{3}\right)+\left(2x^{3}-4x^{2}y-8xy^{2}+5y^{3}\right)\\&\quad=\left(x^{3}+2x^{3}\right)+\left(-2x^{2}y-4x^{2}y\right)+\left(3xy^{2}-8xy^{2}\right)+\left(y^{3}+5y^{3}\right)\\&\quad=3x^{3}-6x^{2}y-5xy^{2}+6y^{3}\end{aligned}$$

However, if we can combine like terms mentally, eliminating the middle step, it is much more efficient to write:

$$\begin{aligned}&\left(x^{3}-2x^{2}y+3xy^{2}+y^{3}\right)+\left(2x^{3}-4x^{2}y-8xy^{2}+5y^{3}\right)\\&\quad=3x^{3}-6x^{2}y-5xy^{2}+6y^{3}\end{aligned}$$

Exercise 5.4.3

Simplify: $\left(-5a^{2}b+4ab-3ab^{2}\right)+\left(2a^{2}b+7ab-ab^{2}\right)$

Answer

$-3a^{2}b+11ab-4ab^{2}d$

Negating a Polynomial

Before attempting subtraction of polynomials, let's first address how to negate or "take the opposite" of a polynomial. First recall that negating is equivalent to multiplying by -1.

Negating

If a is any number, then

$$-a=(-1)a.$$

That is, negating is equivalent to multiplying by -1.

We can use this property to simplify $-(a+b)$. First, negating is identical to multiplying by -1. Then we can distribute the -1.

$$\begin{aligned}-(a+b)&=(-1)(a+b) &&\text{Negating is equivalent to multiplying by } -1\\&=(-1)a+(-1)b &&\text{Distribute the } -1.\\&=-a+(-b) &&\text{Simplify: } (-1)a=-a \text{ and } (-1)b=-b\\&=-a-b &&\text{Subtraction means add the opposite.}\end{aligned}$$

Thus, $-(a+b)=-a-b$. However, it is probably simpler to note that the minus sign in front of the parentheses simply changed the sign of each term inside the parentheses.

Negating a sum

When negating a sum of terms, the effect of the minus sign is to change each term in the parentheses to the opposite sign.

$$-(a+b)=-a-b$$

Let's look at this principle in the next example.

Example 5.4.4

Simplify: $-(-3x^2+4x-8)$

Solution

First, negating is equivalent to multiplying by -1. Then distribute the -1.

$$\begin{aligned} & -(-3x^2+4x-8) \\ & =(-1)\left(-3x^2+4x-8\right) \quad \text{Negating is equivalent to multiplying by } -1 \\ & =(-1)\left(-3x^2\right)+(-1)(4x)-(-1)(8) \quad \text{Distribute the } -1 \\ & =3x^2+(-4x)-(-8) \quad \text{Simplify: } (-1)(-3x^2)=3x^2, (-1)(4x)=-4x, \text{and} (-1)(8)=-8 \\ & =3x^2-4x+8 \quad \text{Subtraction means add the opposite.} \end{aligned}$$

Alternate solution:

As we saw above, a negative sign in front of a parentheses simply changes the sign of each term inside the parentheses. So it is much more efficient to write

$$-(-3x^2+4x-8)=3x^2-4x+8$$

simply changing the sign of each term inside the parentheses.

Exercise 5.4.4

Simplify: $-(2x^2-3x+9)$

Answer

$-2x^2+3x-9$

Subtracting Polynomials

Now that we know how to negate a polynomial (change the sign of each term of the polynomial), we're ready to subtract polynomials.

Example 5.4.5

Simplify: $(y^3-3y^2z+4yz^2+z^3)-(2y^3-8y^2z+2yz^2-8z^3)$

Solution

First, distribute the minus sign, changing the sign of each term of the second polynomial.

$$\begin{aligned} & \left(y^3-3y^2z+4yz^2+z^3\right)-\left(2y^3-8y^2z+2yz^2-8z^3\right) \\ & =y^3-3y^2z+4yz^2+z^3-2y^3+8y^2z-2yz^2+8z^3 \end{aligned}$$

Regroup, combining like terms. You may perform this next step mentally if you wish.

$$\begin{aligned} & =\left(y^3-2y^3\right)+\left(-3y^2z+8y^2z\right)+\left(4yz^2-2yz^2\right)+\left(z^3+8z^3\right) \\ & =-y^3+5y^2z+2yz^2+9z^3 \end{aligned}$$

Exercise 5.4.5

Simplify: $\left(4a^2b+2ab-7ab^2\right)-\left(2a^2b-ab-5ab^2\right)$

Answer

$2a^2b+3ab-2ab^2$

Let's subtract two polynomial functions.

Example 5.4.6

Given $p(x)=-5x^3+6x-9$ and $q(x)=6x^2-7x-11$, simplify $p(x)-q(x)$.

Solution

First, replace $p(x)$ and $q(x)$ with their definitions. Because we are asked to subtract all of $q(x)$ from all of $p(x)$, it is critical to surround each polynomial with parentheses.

$$p(x)-q(x)=\left(-5x^3+6x-9\right)-\left(6x^2-7x-11\right)$$

Distribute the minus sign, changing the sign of each term in the second polynomial, then regroup and combine like terms.

$$\begin{aligned}&=-5x^3+6x-9-6x^2+7x+11\\&={\color{red}-5x^3-6x^2+(6x+7x)+(-9+11)}\\&=-5x^3-6x^2+13x+2\end{aligned}$$

However, after distributing the minus sign, if we can combine like terms mentally, eliminating the middle step, it is much more efficient to write:

$$\begin{aligned}p(x)-q(x)&=\left(-5x^3+6x-9\right)-\left(6x^2-7x-11\right)\\&=-5x^3+6x-9-6x^2+7x+11\\&=-5x^3-6x^2+13x+2\end{aligned}$$

Exercise 5.4.6

Given $f(x)=3x^2+9x-4$ and $g(x)=-5x2+4x-6$, simplify $f(x)-g(x)$.

Answer

$8x^2+5x+2$

Some Applications

Recall that the area of a rectangle having length L and width W is found using the formula $A=LW$. The area of a square having side s is found using the formula $A=s^2$ (see Figure 5.4.1).

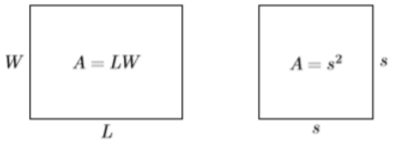


Figure 5.4.1: Area formulae for the rectangle and square.

Example 5.4.7

Find the area of the square in Figure 5.4.2 by summing the area of its parts.

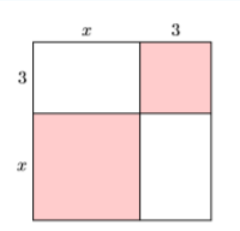


Figure 5.4.2: Find the sum of the parts.

Solution

Let's separate each of the four pieces and label each with its area (see Figure 5.4.3).

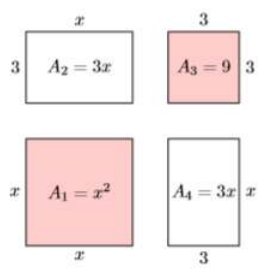


Figure 5.4.3: Finding the area of each of the four parts.

The two shaded squares in Figure 5.4.3 have areas $A_1 = x^2$ and $A_3 = 9$, respectively. The two unshaded rectangles in Figure 5.4.3 have areas $A_2 = 3x$ and $A_4 = 3x$. Summing these four areas gives us the area of the entire figure.

$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 \\ &= x^2 + 3x + 9 + 3x \\ &= x^2 + 6x + 9 \end{aligned}$$

Exercise 5.4.7

Find the area of the square shown below by summing the area of its parts.

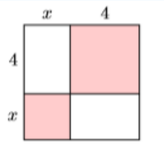


Figure 5.4.4

Answer

$x^2 + 8x + 16$

Example 5.4.8

Ginger runs a business selling wicker baskets. Her business costs for producing and selling x wicker baskets are given by the polynomial function $C(x) = 100 + 3x - 0.02x^2$. The revenue she earns from selling x wicker baskets is given by the polynomial function $R(x) = 2.75x$. Find a formula for $P(x)$, the profit made from selling x wicker baskets. Use your formula to determine Ginger's profit if she sells 123 wicker baskets.

Solution

The profit made from selling x wicker baskets is found by subtracting the costs incurred from the revenue received. In symbols:

$$P(x) = R(x) - C(x)$$

Next, replace $R(x)$ and $C(x)$ with their definitions. Because we are supposed to subtract all of the cost from the revenue, be sure to surround the cost polynomial with parentheses.

$$P(x) = 2.75x - (100 + 3x - 0.02x^2)$$

Distribute the minus sign and combine like terms.

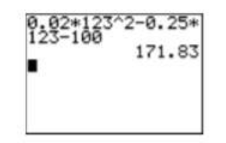
$$\begin{aligned} &= 2.75x - 100 - 3x + 0.02x^2 \\ &= 0.02x^2 - 0.25x - 100 \end{aligned}$$

Thus, the profit function is $P(x)=0.02x^2-0.25x-100$.

Next, to determine the profit if 123 wicker baskets are sold, substitute 123 for x in the profit function $P(x)$.

$$\begin{aligned} P(x) &= 0.02x^2-0.25x-100 \\ P(123) &= 0.02(123)^2-0.25(123)-100 \end{aligned}$$

You can now use your graphing calculator to determine the profit (see Figure $5.4.5$). Hence, the profit made from selling 123 wicker baskets is $\$171.83$.



```
0.02*123^2-0.25*
123-100
                171.83
```

Figure $5.4.5$: Determining the profit from selling 123 wicker baskets.

Exercise $5.4.8$

The costs for producing and selling x widgets are given by the polynomial function $C(x)=50+5x-0.5x^2$, and the revenue for selling x widgets is given by the polynomial function $R(x)=3.5x$. Determine the profit if 75 widgets are sold.

Answer

$\$2,650$

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5.5: Laws of Exponents

In Chapter 1, section 1, we first introduced the definition of an exponent. For convenience, we repeat that definition.

In the exponential expression a^n, the number a is called the base, while the number n is called the exponent.

Exponents

Let a be any real number and let n be any whole number. If $n \neq 0$, then:

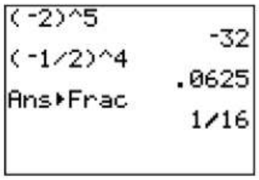
$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

That is, to calculate a^n, write a as a factor n times. In the case where $a \neq 0$, but $n = 0$, then we define:

$$a^0 = 1$$

For example, raising a number to the fifth power requires that we repeat the number as a factor five times (see Figure $5.5.1$).

$$\begin{aligned} (-2)^5 &= (-2)(-2)(-2)(-2)(-2) \\ &= -32 \end{aligned}$$



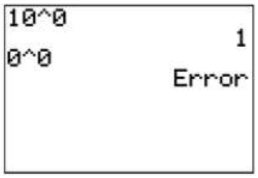
```
(-2)^5
                -32
(-1/2)^4
              .0625
Ans▸Frac
               1/16
```

Figure $5.5.1$: Evaluating $(-2)^5$ and $(-1/2)^4$. Recall: Use the **MATH** key to locate the **►Frac** command.

Raising a number to the fourth power requires that we repeat that number as a factor four times (see Figure $5.5.1$).

$$\begin{aligned} \left(-\frac{1}{2}\right)^4 &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) \\ &= \frac{1}{16} \end{aligned}$$

As a final example, note that $10^0 = 1$, but 0^0 is undefined (see Figure $5.5.2$).



```
10^0
                  1
0^0
              Error
```

Figure $5.5.2$: Evaluating 10^0 and 0^0 on the graphing calculator.

Note

For those who may be wondering why $a^0 = 1$, provided $a \neq 0$, here is a nice argument. First, note that $a^1 = a$, so:

$$a \cdot a^0 = a^1 \cdot a^0$$

On the right, repeat the base and add the exponents.

$$a \cdot a^0 = a^1$$

Or equivalently:

$$a \cdot a^0 = a$$

Now, divide both sides by a, which is permissible if $a \neq 0$.

$$\frac{a \cdot a^0}{a} = \frac{a}{a}$$

Simplify both sides:

$$a^0 = 1$$

Multiplying With Like Bases

In the expression a^n, the number a is called the base and the number n is called the *exponent*. Frequently, we'll be required to multiply two exponential expressions with like bases, such as $x^3 \cdot x^4$. Recall that the exponent tells us how many times to write each base as a factor, so we can write:

$$\begin{aligned} x^3 \cdot x^4 &= (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \\ &= x^7 \end{aligned}$$

Note that we are simply counting the number of times that x occurs as a factor. First, we have three $x's$, then four $x's$, for a total of seven $x's$. However a little thought tells us that it is much quicker to simply add the exponents to reveal the total number of times x occurs as a factor.

$$\begin{aligned} x^3 \cdot x^4 &= x^{3+4} \\ &= x^7 \end{aligned}$$

The preceding discussion is an example of the following general law of exponents.

Multiplying With Like Bases

To multiply two exponential expressions with like bases, repeat the base and add the exponents.

$$a^m \cdot a^n = a^{m+n}$$

Example 5.5.1

Simplify each of the following expressions:

a. $y^4 \cdot y^8$
b. $2^3 \cdot 2^5$
c. $(x+y)^2(x+y)^7$

Solution

In each example we have like bases. Thus, the approach will be the same for each example: repeat the base and add the exponents.

a. $$\begin{aligned} y^4 \cdot y^8 &= y^{4+8} \\ &= y^{12} \end{aligned}$$
b. $$\begin{aligned} 2^3 \cdot 2^5 &= 2^{3+5} \\ &= 2^8 \end{aligned}$$
c. $$\begin{aligned} (x+y)^2(x+y)^7 &= (x+y)^{2+7} \\ &= (x+y)^9 \end{aligned}$$

With a little practice, each of the examples can be simplified mentally. Repeat the base and add the exponents in your head: $y^4 \cdot y^8 = y^{12}$, $2^3 \cdot 2^5 = 2^8$ and $(x+y)^2(x+y)^7 = (x+y)^9$.

Exercise 5.5.1

$3^4 \cdot 3^2$

Answer

3^6

Example 5.5.2

Simplify: $\left(a^6b^4\right)\left(a^3b^2\right)$

Solution

We'll use the commutative and associative properties to change the order of operation, then repeat the common bases and add the exponents.

$$\begin{aligned}\left(a^6b^4\right)\left(a^3b^2\right) &= a^6b^4a^3b^2 \\ &\qquad \text{The associative property allows us to regroup in the order we prefer.} \\ &= a^6a^3b^4b^2 \\ &\qquad \text{The commutative property allows us to change the order of multiplication.} \\ &= a^9b^6 \quad \text{Repeat the common bases and add the exponents.}\end{aligned}$$

With practice, we realize that if all of the operators are multiplication, then we can multiply in the order we prefer, repeating the common bases and adding the exponents mentally: $\left(a^6b^4\right)\left(a^3b^2\right) = a^9b^6$.

Exercise 5.5.2

$\left(x^2y^6\right)\left(x^4y^3\right)$

Answer

x^6y^9

Example 5.5.3

Simplify: $x^{n+3} \cdot x^{3-2n}$

Solution

Again, we repeat the base and add the exponents.

$$\begin{aligned}x^{n+3} \cdot x^{3-2n} &= x^{(n+3)+(3-2n)} \quad \text{Repeat the base, add the exponents.} \\ &= x^{6-n} \quad \text{Simplify. Combine like terms.}\end{aligned}$$

Exercise 5.5.3

$x^{5-n} \cdot x^{4n+2}$

Answer

x^{3n+7}

Dividing With Like Bases

Like multiplication, we will also be frequently asked to divide exponential expressions with like bases, such as x^7/x^4. Again, the key is to remember that the exponent tells us how many times to write the base as a factor, so we can write:

$$\begin{aligned}\frac{x^7}{x^4} &= \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x} \\ &= \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} \\ &= x^3\end{aligned}$$

Note how we cancel four $x's$ in the numerator for four $x's$ in the denominator. However, in a sense we are "subtracting four $x's$" from the numerator, so a faster way to proceed is to repeat the base and subtract the exponents, as follows:

$$\frac{x^7}{} = x^{7-4}$$

$$x^4$$

$$= x^3$$

The preceding discussion is an example of the second general law of exponents.

How to Divide with Like Bases

To divide two exponential expressions with like bases, repeat the base and subtract the exponents. Given $a \neq 0$,

$$\frac{a^m}{a^n} = a^{m-n}$$

Note that the subtraction of the exponents follows the rule “top minus bottom.”

Note

Here is another nice argument why $a^0 = 1$, provided $a \neq 0$. Start with:

$$\frac{a^1}{a^1} = 1$$

Repeat the base and subtract the exponents.

$$a^{1-1} = 1$$

Simplify.

$$a^0 = 1$$

Example 5.5.4

Simplify each of the following expressions:

a. $\dfrac{x^{12}}{x^3}$
b. $\dfrac{5^7}{5^7}$
c. $\dfrac{(2x+1)^8}{(2x+1)^3}$

Solution

In each example we have like bases. Thus, the approach will be the same for each example: repeat the base and subtract the exponents.

a. $$\begin{aligned}\frac{x^{12}}{x^3} &= x^{12-3} \\ &= x^9\end{aligned}$$
b. $$\begin{aligned}\frac{5^7}{5^7} &= 5^{7-7} \\ &= 5^0 \\ &= 1\end{aligned}$$
c. $$\begin{aligned}\frac{(2x+1)^8}{(2x+1)^3} &= (2x+1)^{8-3} \\ &= (2x+1)^5\end{aligned}$$

With a little practice, each of the examples can be simplified mentally. Repeat the base and subtract the exponents in your head: $x^{12}/x^3 = x^9, 5^7/5^4 = 5^3$ and $(2x+1)^8/(2x+1)^3 = (2x+1)^5$.

Exercise 5.5.4

$$\frac{4^5}{4^3}$$

Answer

4^2

Example 5.5.5

Simplify: $\frac{12x^5y^7}{4x^3y^2}$

Solution

We first express the fraction as a product of three fractions, the latter two with a common base. In the first line of the following solution, note that if you multiply numerators and denominators of the three separate fractions, the product equals the original fraction on the left.

$$\begin{aligned} \frac{12x^5y^7}{4x^3y^2} &= \frac{12}{4} \cdot \frac{x^5}{x^3} \cdot \frac{y^7}{y^2} && \text{Break into a product of three fractions.} \\ &= 3x^{5-3}y^{7-2} && \text{Simplify: } 12/4 = 3. \text{ Then repeat the common} \\ &= 3x^2y^5 && \text{Simplify.} \end{aligned}$$

Exercise 5.5.5

Simplify: $\frac{15a^6b^9}{3ab^5}$

Answer

$5a^5b^4$

Example 5.5.6

Simplify: $\frac{x^{5n-4}}{x^{3-2n}}$

Solution

Again, we repeat the base and subtract the exponents.

$$\begin{aligned} \frac{x^{5n-4}}{x^{3-2n}} &= x^{(5n-4)-(3-2n)} && \text{Repeat the base, subtract exponents.} \\ &= x^{5n-4-3+2n} && \text{Distribute the minus sign.} \\ &= x^{7n-7} && \text{Simplify. Combine like terms.} \end{aligned}$$

Exercise 5.5.6

Simplify: $\frac{x^{3n-6}}{x^{n+2}}$

Answer

x^{2n-8}

Raising a Power to a Power

Suppose we have an exponential expression raised to a second power, such as $(x^2)3$. The second exponent tells us to write x^2 as a factor three times:

$$\begin{aligned} (x^2)^3 &= x^2 \cdot x^2 \cdot x^2 && \text{Write } x^2 \text{ as a factor three times.} \\ &= x^6 && \text{Repeat the base, add the exponents.} \end{aligned}$$

Note how we added $2+2+2$ to get 6. However, a much faster way to add "three twos" is to multiply: $3 \cdot 2 = 6$. Thus, when raising a "power to a second power," repeat the base and multiply the exponents, as follows:

$$\begin{aligned} (x^2)^3 &= x^{2 \cdot 3} \\ &= x^6 \end{aligned}$$

The preceding discussion gives rise to the following third law of exponents.

Raising a Power to a Power

When raising a power to a power, repeat the base and multiply the exponents. In symbols:

$$(a^m)^n = a^{mn}$$

Note that juxtaposing two variables, as in mn, means “m times n.”

Example 5.5.7

Simplify each of the following expressions:

a. $\left(z^3\right)^5$
b. $\left(7^3\right)^0$
c. $\left[(x-y)^3\right]^6$

Solution

In each example we are raising a power to a power. Hence, in each case, we repeat the base and multiply the exponents.

a. $\begin{aligned}\left(z^3\right)^5 &= z^{3.5} \\ &= z^{15}\end{aligned}$

b. $\begin{aligned}\left(7^3\right)^0 &= 7^{3.0} \\ &= 7^0 \\ &= 1\end{aligned}$

c. $\begin{aligned}\left[(x-y)^3\right]^6 &= (x-y)^{3\cdot 6} \\ &= (x-y)^{18}\end{aligned}$

With a little practice, each of the examples can be simplified mentally. Repeat the base and multiply the exponents in your head: $\left(z^3\right)^5 = z^{15}, \left(7^3\right)^4 = 7^{12}$ and $\left[(x-y)^3\right]^6 = (x-y)^{18}$.

Exercise 5.5.7

Simplify: $\left(2^3\right)^4$

Answer

2^{12}

Example 5.5.8

Simplify: $\left(x^{2n-3}\right)^4$

Solution

Again, we repeat the base and multiply the exponents.

$$\begin{aligned}\left(x^{2n-3}\right)^4 &= x^{4(2n-3)} && \text{Repeat the base, multiply exponents.} \\ &= x^{8n-12} && \text{Distribute the 4.}\end{aligned}$$

Exercise 5.5.8

Simplify: $\left(a^{2-n}\right)^3$

Answer

a^{6-3n}

Raising a Product to a Power

We frequently have need to raise a product to a power, such as $(xy)^3$. Again, remember the exponent is telling us to write xy as a factor three times, so:

$$\begin{aligned}(xy)^3 &= (xy)(xy)(xy) && \text{Write } xy \text{ as a factor three times.}\\ &= xyxyxy && \text{The associative property allows us to group as we please.}\\ &= xxxyyy && \text{The commutative property allows us to change the order as we please.}\\ &= x^3y^3 && \text{Invoke the exponent definition: } xxx = x^3 \text{ and } yyy = y^3\end{aligned}$$

However, it is much simpler to note that when you raise a product to a power, you raise each factor to that power. In symbols:

$$(xy)^3 = x^3y^3$$

The preceding discussion leads us to a fourth law of exponents.

Raising a Product to a Power

To raise a product to a power, raise each factor to that power. In symbols:

$$(ab)^n = a^nb^n$$

Example 5.5.9

Simplify each of the following expressions:

a. $(yz)^5$
b. $(-2x)^3$
c. $(-3y)^2$

Solution

In each example we are raising a product to a power. Hence, in each case, we raise each factor to that power.

a. $(yz)^5 = y^5z^5$
b. $$\begin{aligned}(-2x)^3 &= (-2)^3x^3\\ &= -8x^3\end{aligned}$$
c. $$\begin{aligned}(-3y)^2 &= (-3)^2y^2\\ &= 9y^2\end{aligned}$$

With a little practice, each of the examples can be simplified mentally. Raise each factor to the indicated power in your head: $(yz)^5 = y^5z^5$, $(-2x)^3 = -8x^3$ and $(-3y)^2 = 9y^2$

Exercise 5.5.9

Simplify: $(-2b)^4$

Answer

$16b^4$

When raising a product of three factors to a power, it is easy to show that we should raise each factor to the indicated power. For example, $(abc)^3 = a^3b^3c^3$. In general, this is true regardless of the number of factors. When raising a product to a power, raise each of the factors to the indicated power.

Example 5.5.10

Simplify: $(-2a^3b^2)^3$

Solution

Raise each factor to the third power, then simplify.

$(-2a^3$

$$b^2)^3 = (-2)^3 \left(a^3\right)^3 \left(b^2\right)^3 \quad \text{Raise each factor to the third power.}$$

$$= -8a^9b^6 \quad \text{Simplify: } (-2)^3 = 8. \text{ In the remaining factors, raising a power to a power requires that we multiply the exponents.}$$

Exercise 5.5.10

Simplify: $(-3xy^4)^5$

Answer

$-243x^5y^{20}$

Example 5.5.11

Simplify: $\left(-2x^2y\right)^2 \left(-3x^3y\right)$

Solution

In the first grouped product, raise each factor to the second power.

$$\begin{aligned}
\left(-2x^2y\right)^2 \left(-3x^3y\right) &= \left((-2)^2 \left(x^2\right)^2 y^2\right) \left(-3x^3y\right) \\
&\quad \text{Raise each factor in the first grouped product to the second power.} \\
&= \left(4x^4y^2\right)\left(-3x^3y\right) \quad \text{Simplify: } (-2)^2 = 4 \text{ and } \left(x^2\right)^2 = x^4
\end{aligned}$$

The associative and commutative property allows us to multiply all six factors in the order that we please. Hence, we’ll multiply 4 and -3, then x^4 and x^3, and \(y^2 and y, in that order. In this case, we repeat the base and add the exponents.

$$= -12x^7y^3 \quad \text{Simplify: } (4)(-3) = -12. \text{ Also, } x^4x^3 = x^7 \text{ and } y^2y = y^3$$

Exercise 5.5.11

Simplify: $\left(-a^3b^2\right)^3 \left(-2a^2b^4\right)^2$

Answer

$-4a^{13}b^{14}$

Raising a Quotient to a Power

Raising a quotient to a power is similar to raising a product to a power. For example, raising $(x/y)^3$ requires that we write x/y as a factor three times.

$$\begin{aligned}
\left(\frac{x}{y}\right)^3 &= \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \\
&= \frac{x \cdot x \cdot x}{y \cdot y \cdot y} \\
&= \frac{x^3}{y^3}
\end{aligned}$$

However, it is much simpler to realize that when you raise a quotient to a power, you raise both numerator and denominator to that power. In symbols:

$$\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$$

This leads to the fifth and final law of exponents.

Raising a Quotient to a Power

To raise a quotient to a power, raise both numerator and denominator to that power. Given $b \neq 0$,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example $5.5.12$

Simplify each of the following expressions:

a. $\left(\frac{2}{3}\right)^2$
b. $\left(\frac{x}{3}\right)^3$
c. $\left(-\frac{2}{y}\right)^4$

Solution

In each example we are raising a quotient to a power. Hence, in each case, we raise both numerator and denominator to that power.

a. $$\begin{aligned}\left(\frac{2}{3}\right)^2 &= \frac{2^2}{3^2} \\ &= \frac{4}{9}\end{aligned}$$

b. $$\begin{aligned}\left(\frac{x}{3}\right)^3 &= \frac{x^3}{3^3} \\ &= \frac{x^3}{27}\end{aligned}$$

c. $$\begin{aligned}\left(-\frac{2}{y}\right)^4 &= \frac{2^4}{y^4} \\ &= \frac{16}{y^4}\end{aligned}$$

Note that in example (c), raising a negative base to an even power produces a positive result. With a little practice, each of the examples can be simplified mentally. Raise numerator and denominator to the indicated power in your head: $(2/3)^2 = 4/9$, $(x/3)^3 = x^3/27$, and $(-2/y)^4 = 16/y^4$

Exercise $5.5.12$

Simplify: $\left(\frac{5}{4}\right)^3$

Answer

$$\frac{125}{64}$$

Example $5.5.13$

Simplify: $\left(\frac{2x^5}{y^3}\right)^2$

Solution

Raise both numerator and denominator to the second power, then simplify:

$$\left(\frac{2x^5}{y^3}\right)^2 = \frac{\left(2x^5\right)^2}{\left(y^3\right)^2} \quad \text{Raise numerator and denominator to the second power.}$$

In the numerator, we need to raise each factor of the product to the second power. Then we need to remind ourselves that when we raise a power to a power, we multiply the exponents.

$$= \frac{2^2\left(x^5\right)^2}{\left(y^3\right)^2} \quad \text{Raise each factor in the numerator and denominator to the second power.}$$

$$= \frac{4x^{10}}{y^6} \quad \text{Simplify: } 2^2 = 4,\ (x^5)^2 = x^{10}, \text{ and } (y^3)^2 = y^6$$

Exercise 5.5.13

Simplify: $\left(\frac{a^4}{3b^2}\right)^3$

Answer

$$\frac{a^{12}}{27b^6}$$

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5.6: Multiplying Polynomials

In this section we will find the products of polynomial expressions and functions. We start with the product of two monomials, then graduate to the product of a monomial and polynomial, and complete the study by finding the product of any two polynomials.

The Product of Monomials

As long as all of the operations are multiplication, we can use the commutative and associative properties to change the order and regroup.

Example 5.6.1

Multiply: $-5(7y)$. Simplify: $-5(7y)$.

Solution

Use the commutative and associative properties to change the order and regroup.

$$\begin{aligned} -5(7y) &= [(-5)(7)]y && \text{Reorder. Regroup.} \\ &= -35y && \text{Multiply: } (-5)(7)=-35 \end{aligned}$$

Exercise 5.6.1

Multiply: $3(4x)$

Answer

$12x$

Example 5.6.2

Multiply: $(-3x^2)(4x^3)$. Simplify: $(-3x^2)(4x^3)$.

Solution

Use the commutative and associative properties to change the order and regroup.

$$\begin{aligned} \left(-3x^2\right)\left(4x^3\right) &= [(-3)(4)]\left(x^2x^3\right) && \text{Reorder. Regroup.} \\ &= -12x^5 && \text{Multiply: } (-3)(4)=12, x^2x^3=x^5 \end{aligned}$$

Exercise 5.6.2

Multiply: $(7y^5)(-2y^2)$

Answer

$-14y^7$

Example 5.6.3

Multiply: $(-2a^2b^3)(-5a^3b)$. Simplify: $(-2a^2b^3)(-5a^3b)$.

Solution

Use the commutative and associative properties to change the order and regroup.

$$\begin{aligned} \left(-2a^2b^3\right)\left(-5a^3b\right) &= [(-2)(-5)]\left(a^2a^3\right)\left(b^3b\right) && \text{Reorder. Regroup.} \\ &= 10a^5b^4 && \text{Multiply: } (-2)(-5)=10, a^2a^3=a^5, \text{ and } b^3b=b^4 \end{aligned}$$

Exercise 5.6.3

Multiply: $(-6st^2)(3s^3t^4)$

Answer

$-18s^4t^6$

When multiplying monomials, it is much more efficient to make the required calculations mentally. In the case of Example 5.6.1, multiply -5 and 7 mentally to get

$$-5(7y) = -35y$$

In the case of Example 5.6.2, multiply -3 and 4 to get -12, then repeat the base x and add exponents to get

$$(-3x^2)(4x^3) = -12x^5$$

Finally, in the case of Example 5.6.3, multiply -2 and -5 to get 10, then repeat the bases and add their exponents.

$$(-2a^2b^3)(-5a^3b) = 10a^5b^4$$

Multiplying a Monomial and a Polynomial

Now we turn our attention to the product of a monomial and a polynomial.

Example 5.6.4

Multiply: $5\left(3x^2 - 4x - 8\right)$

Solution

We need to first distribute the 5 times each term of the polynomial. Then we multiply the resulting monomials mentally.

$$\begin{aligned} 5\left(3x^2 - 4x - 8\right) &= 5\left(3x^2\right) - 5(4x) - 5(8) \\ &= 15x^2 - 20x - 40 \end{aligned}$$

Exercise 5.6.4

Multiply: $4\left(-2y^2 + 3y + 5\right)$

Answer

$-8y^2 + 12y + 20$

Example 5.6.5

Multiply: $2y(-3y + 5)$

Solution

We need to first distribute the $2y$ times each term of the polynomial. Then we multiply the resulting monomials mentally.

$$\begin{aligned} 2y(-3y + 5) &= 2y(-3y) + 2y(5) \\ &= -6y^2 + 10y \end{aligned}$$

Exercise 5.6.5

Multiply: $-3x(4 - 2x)$

Answer

$-12x + 6x^2$

Example 5.6.6

Multiply: $-3ab\left(a^2+2ab-b^2\right)$

Solution

We need to first distribute the $-3ab$ times each term of the polynomial. Then we multiply the resulting monomials mentally.

$$\begin{aligned}-3ab\left(a^2+2ab-b^2\right)&=-3ab\left(a^2\right)+(-3ab)(2ab)-(-3ab)\left(b^2\right)\\&=-3a^3b+\left(-6a^2b^2\right)-\left(-3ab^3\right)\\&=-3a^3b-6a^2b^2+3ab^3\end{aligned}$$

Alternate solution:

It is far more efficient to perform most of the steps of the product $-3ab(a^2+2ab-b^2)$ mentally. We know we must multiply $-3ab$ times each of the terms of the polynomial $a^2+2ab-b^2$. Here are our mental calculations:

i. $-3ab$ times a^2 equals $-3a3b$.
ii. $-3ab$ times $2ab$ equals $-6a^2b^2$.
iii. $-3ab$ times $-b^2$ equals $3ab^3$.

Thinking in this manner allows us to write down the answer without any of the steps shown in our first solution. That is, we immediately write:

$$-3ab\left(a^2+2ab-b^2\right)=-3a^3b-6a^2b^2+3ab^3$$

Exercise 5.6.6

Multiply: $2xy\left(x^2-4xy^2+7x\right)$

Answer

$2x^3y-8x^2y^3+14x^2y$

Example 5.6.7

Multiply: $-2z^2\left(z^3+4z^2-11\right)$

Solution

We need to first distribute the $-2z^2$ times each term of the polynomial. Then we multiply the resulting monomials mentally.

$$\begin{aligned}-2z^2\left(z^3+4z^2-11\right)&=-2z^2\left(z^3\right)+\left(-2z^2\right)\left(4z^2\right)-\left(-2z^2\right)(11)\\&=-2z^5+\left(-8z^4\right)-\left(-22z^2\right)\\&=-2z^5-8z^4+22z^2\end{aligned}$$

Alternate solution:

It is far more efficient to perform most of the steps of the product $-2z^2(z^3+4z^2-11)$ mentally. We know we must multiply $-2z^2$ times each of the terms of the polynomial z^3+4z^2-11. Here are our mental calculations:

i. $-2z^2$ times z^3 equals $-2z^5$.
ii. $-2z^2$ times $4z^2$ equals $-8z^4$.
iii. $-2z^2$ times -11 equals $22z^2$.

Thinking in this manner allows us to write down the answer without any of the steps shown in our first solution. That is, we immediately write:

$$-2z^2\left(z^3+4z^2-11\right)=-2z^5-8z^4+22z^2$$

Exercise 5.6.7

Multiply: $-5y^3\left(y^2-2y+1\right)$

Answer

$-5y^5+10y^4-5y^3$

Multiplying Polynomials

Now that we've learned how to take the product of two monomials or the product of a monomial and a polynomial, we can apply what we've learned to multiply two arbitrary polynomials.

Before we begin with the examples, let's revisit the distributive property. We know that

$$2\cdot(3+4)=2\cdot3+2\cdot4$$

Both sides are equal to 14. We're used to having the monomial factor on the left, but it can also appear on the right.

$$(3+4)\cdot2=3\cdot2+4\cdot2$$

Again, both sides equal 14. So, whether the monomial appears on the left or right makes no difference; that is, $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ give the same result. In each case, you multiply a times both terms of $b+c$.

Example 5.6.8

Multiply: $(2x+5)(3x+2)$

Solution

Note that $(2x+5)(3x+2)$ has the form $(b+c)a$, where a is the binomial $3x+2$. Because $(b+c){\color{red}a}=b{\color{red}a}+c{\color{red}a}$, we will multiply $3x+2$ times both terms of $2x+5$ in the following manner.

$$(2x+5){\color{red}(3x+2)}=2x{\color{red}(3x+2)}+5{\color{red}(3x+2)}$$

Now we distribute monomials times polynomials, then combine like terms.

$$\begin{aligned}&=6x^2+4x+15x+10\\&=6x^2+19x+10\end{aligned}$$

Thus, $(2x+5)(3x+2)=6x^2+19x+10$

Exercise 5.6.8

Multiply: $(3x+4)(5x+1)$

Answer

$15x^2+23x+4$

Example 5.6.9

Multiply: $(x+5)\left(x^2+2x+7\right)$

Solution

Note that $(x+5)(x^2+2x+7)$ has the form $(b+c)a$, where a is the trinomial x^2+2x+7. Because $(b+c){\color{red}a}=b{\color{red}a}+c{\color{red}a}$, we will multiply x^2+2x+7 times both terms of $x+5$ in the following manner.

$$(x+5){\color{red}\left(x^2+2x+7\right)}=x{\color{red}\left(x^2+2x+7\right)}+5{\color{red}\left(x^2+2x+7\right)}$$

Now we distribute monomials times polynomials, then combine like terms.

$$\begin{aligned}&=x^3+2x^2+7x+5x^2+10x+35\\&=x^3+7x^2+17x+35\end{aligned}$$

Thus, $(x+5)\left(x^2+2x+7\right)=x^3+7x^2+17x+35$

Exercise 5.6.9

Multiply: $(z+4)\left(z^2+3z+9\right)$

Answer

$z^3+7z^2+21z+36$

Speeding Things Up a Bit

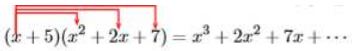
Let's re-examine Example 5.6.9 with the hope of unearthing a pattern that will allow multiplication of polynomials to proceed more quickly with less work. Note the first step of Example 5.6.9.

$$(x+5)\left(x^2+2x+7\right)=x\left(x^2+2x+7\right)+5\left(x^2+2x+7\right)$$

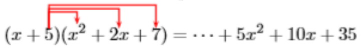
Note that the first product on the right is the result of taking the product of the first term of $x+5$ and x^2+2x+7. Similarly, the second product on the right is the result of taking the product of the second term of $x+5$ and x^2+2x+7. Next, let's examine the result of the second step.

$$(x+5)\left(x^2+2x+7\right)=x^3+2x^2+7x+5x^2+10x+35$$

The first three terms on the right are the result of multiplying x times x^2+2x+7.



The second set of three terms on the right are the result of multiplying 5 times x^2+2x+7.



These notes suggest an efficient shortcut. To multiply $x+5$ times x^2+2x+7,

- Multiply x times each term of x^2+2x+7.
- Multiply 5 times each term of x^2+2x+7.
- Combine like terms.

This process would have the following appearance.

$$\begin{aligned}(x+5)\left(x^2+2x+7\right)&=x^3+2x^2+7x+5x^2+10x+35\\&=x^3+7x^2+17x+35\end{aligned}$$

Example 5.6.10

Use the shortcut technique described above to simplify: $(2y-6)\left(3y^2+4y+11\right)$

Solution

Multiply $2y$ times each term of $3y^2+4y+11$, then multiply -6 times each term of $3y^2+4y+11$. Finally, combine like terms.

$$\begin{aligned}(2y-6)\left(3y^2+4y+11\right)&=6y^3+8y^2+22y-18y^2-24y-66\\&=6y^3-10y^2-2y-66\end{aligned}$$

Exercise 5.6.10

Multiply: $(3x+2)\left(4x^2-x+10\right)$

Answer

$12x^3+5x^2+28x+20$

Example 5.6.11

Use the shortcut technique to simplify $(a+b)^2$

Solution

To simplify $(a+b)^2$, we first must write $a+b$ as a factor two times.

$$(a+b)^2=(a+b)(a+b)$$

Next, multiply a times both terms of $a+b$, multiply b times both terms of $a+b$, then combine like terms.

$$\begin{aligned}&=a^2+ab+ba+b^2\\&=a^2+2ab+b^2\end{aligned}$$

Note that $ab=ba$ because multiplication is commutative, so $ab+ba=2ab$.

Exercise 5.6.11

Multiply: $(3y-2)^2$

Answer

$9y^2-12y+4$

Example 5.6.12

Use the shortcut technique to simplify: $\left(x^2+x+1\right)\left(x^2-x-1\right)$

Solution

This time the first factor contains three terms, x^2, x, and 1, so we first multiply x^2 times each term of x^2-x-1, then x times each term of x^2-x-1, and 1 times each term of x^2-x-1. Then we combine like terms.

$$\begin{aligned}\left(x^2+x+1\right)\left(x^2-x-1\right)&=x^4-x^3-x^2+x^3-x^2-x+x^2-x-1\\&=x^4-x^2-2x-1\end{aligned}$$

Exercise 5.6.12

Multiply: $\left(a^2-2a+3\right)\left(a^2+2a-3\right)$

Answer

$a^4-4a^2+12a-9$

Some Applications

Recall that the area of a circle of radius r is found using the formula $A=\pi r^2$. The circumference (distance around) of a circle of radius r is found using the formula $C=2\pi r$ (see Figure $5.6.1$).

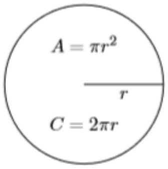


Figure $5.6.1$: Formulae for the area and circumference of a circle of radius r.

Example 5.6.13

In Figure 5.6.2 are pictured two concentric circles (same center). The inner circle has radius x and the outer circle has radius $x+1$. Find the area of the shaded region (called an annulus) as a function of x.

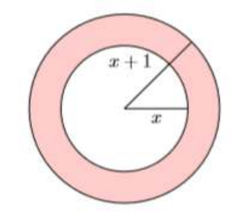


Figure 5.6.2: Find the area of the shaded region.

Solution

We can find the area of the shaded region by subtracting the area of the inner circle from the area of the outer circle.

$$A = \text{ Area of outer circle - Area of inner circle}$$

We use the formula $A=\pi r^2$ to compute the area of each circle. Because the outer circle has radius $x+1$, it has area $\pi(x+1)^2$. Because the inner circle has radius x, it has area πx^2.

$$=\pi(x+1)^2-\pi x^2$$

Next, we'll expand $(x+1)^2$, then combine like terms.

$$\begin{aligned}&=\pi(x+1)(x+1)-\pi x^2\\&=\pi\left(x^2+x+x+1\right)-\pi x^2\\&=\pi\left(x^2+2x+1\right)-\pi x^2\end{aligned}$$

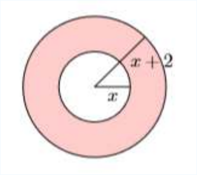
Finally, distribute π times each term of x^2+2x+1, then combine like terms.

$$\begin{aligned}&=\pi x^2+2\pi x+\pi-\pi x^2\\&=2\pi x+\pi\end{aligned}$$

Hence, the area of the shaded region is $A=2\pi x+\pi$.

Exercise 5.6.13

Two concentric circles are shown below. The inner circle has radius x and the outer circle has radius $x+2$. Find the area of the shaded region as a function of x.



Answer

$4\pi x+4\pi$

Example 5.6.14

The demand for widgets is a function of the unit price, where the demand is the number of widgets the public will buy and the unit price is the amount charged for a single widget. Suppose that the demand is given by the function $x=270-0.75p$, where x is the demand and p is the unit price. Note how the demand decreases as the unit price goes up (makes sense). Use the graphing calculator to determine the unit price a retailer should charge for widgets so that his revenue from sales equals $\$20,000$.

Solution

To determine the revenue (R), you multiply the number of widgets sold (x) by the unit price (p).

$$R = xp \tag{5.6.1}$$

However, we know that the number of units sold is the demand, given by the formula

$$x = 270 - 0.75p \tag{5.6.2}$$

Substitute equation 5.6.2 into equation 5.6.1 to obtain the revenue as a function of the unit price.

$$R = (270 - 0.75p)p$$

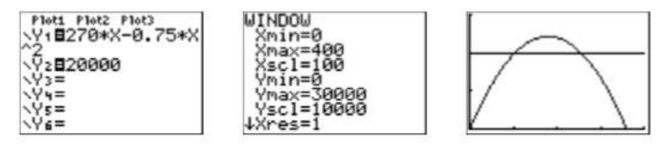
Expand.

$$R = 270p - 0.75p^2 \tag{5.6.3}$$

We're asked to determine the unit price that brings in a revenue of $\$20,000$. Substitute $\$20,000$ for R in equation 5.6.3.

$$20000 = 270p - 0.75p^2 \tag{5.6.4}$$

Enter each side of equation 5.6.4 into the **Y= menu** of your calculator (see the first image in Figure 5.6.3). After some experimentation, we settled on the **WINDOW** parameters shown in the second image of Figure 5.6.3. After making these settings, push the **GRAPH** button to produce the graph shown in the third image in Figure 5.6.3.



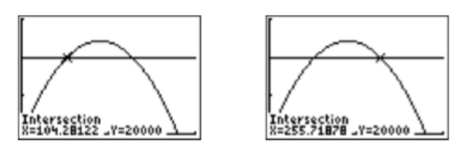
```
\Y1■270*X-0.75*X
^2
\Y2■20000
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=0
 Xmax=400
 Xscl=100
 Ymin=0
 Ymax=30000
 Yscl=10000
↓Xres=1
```

Figure 5.6.3: Solving $20000 = 270p - 0.75p^2$.

Note that the third image in Figure 5.6.3 shows that there are two solutions, that is, two ways we can set the unit price to obtain a revenue of $\$20,000$. To find the first solution, select 5:intersect from the **CALC** menu, press **ENTER** in response to "First curve," press **ENTER** in response to "Second curve," then move your cursor closer to the point of intersection on the left and press **ENTER** in response to "Guess." The result is shown in the first image in Figure 5.6.4.

Repeat the process to find the second point of intersection. The result is shown in the second image in Figure 5.6.4.



```
Intersection
X=104.28122  Y=20000
```

```
Intersection
X=255.71878  Y=20000
```

Figure 5.6.4: Finding the points of intersection.

Note

In this example, the horizontal axis is actually the p-axis. So when we set Xmin and Xmax, we're actually setting bounds on the p-axis.

Reporting the solution on your homework:

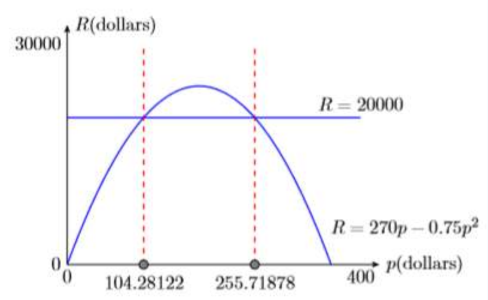


Figure 5.6.5: Reporting your graphical solution on your homework.

Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with p and R, respectively (see Figure 5.6.5). Include the units (dollars and dollars).
- Place your **WINDOW** parameters at the end of each axis (see Figure 5.6.5).
- Label each graph with its equation (see Figure 5.6.5).
- Drop a dashed vertical line through each point of intersection. Shade and label the p-values of the points where the dashed vertical lines cross the p-axis. These are the solutions of the equation $20000 = 270p - 0.75p^2$ (see Figure 5.6.5).

Rounding to the nearest penny, setting the unit price at either $\$104.28$ or $\$255.72$ will bring in a revenue of $\$20,000$.

Exercise 5.6.14

Suppose that the demand for gadgets is given by the function $x = 320 - 1.5p$, where x is the demand and p is the unit price. Use the graphing calculator to determine the unit price a retailer should charge for gadgets so that her revenue from sales equals $\$12,000$.

Answer

$\$48.55$ or $\$164.79$

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5.7: Special Products

This section is dedicated to explaining a number of important shortcuts for multiplying binomials. These are extremely important patterns that will produce the same products computed in previous sections. It is essential that readers practice until they become proficient using each of the patterns presented in this section.

The FOIL Method

Consider the product of two binomials $(x+3)(x+6)$. We already know how to find the product of these two binomials; we multiply x times both terms of $x+6$, then we multiply 3 times both terms of $x+6$.

$$(x+3)(x+6)=x^2+6x+3x+18$$

Normally we combine like terms, but we halt the process at this point so as to introduce the pattern called the FOIL method. The letters in the word FOIL stand for “First,” “Outer,” “Inner,” and “Last.”

Let’s see how we can connect these terms to the product $(x+3)(x+6)$.

- The arrows indicate the terms in the “First” positions in each binomial. If you multiply the terms in the “First” position, you get x^2.

$$(\underbrace{x+3)(x}_{F}+6)$$

- The arrows indicate the terms in the “Outer” positions in each binomial. If you multiply the terms in the “Outer” positions, you get $6x$.

$$(\underbrace{x+3)(x+6}_{O})$$

- The arrows indicate the terms in the “Inner” positions in each binomial. If you multiply the terms in the “Inner” positions, you get $3x$.

$$(x+\underbrace{3)(x}_{I}+6)$$

- The arrows indicate the terms in the “Last” positions in each binomial. If you multiply the terms in the “Last” positions, you get 18.

$$(x+\underbrace{3)(x+6}_{L})$$

The following diagram shows the connection between “First,” “Outer,” “Inner,” “Last,” and the answer.

$$(x+3)(x+6)=\overset{F}{x^2}+\overset{O}{6x}+\overset{I}{3x}+\overset{L}{18}$$

Example 5.7.1

Use the FOIL method to simplify: $(x+5)(x+7)$

Solution

Multiply the “First” positions: x^2. Multiply the “Outer” positions: $7x$. Multiply the “Inner” positions: $5x$. Multiply the “Last” positions: 35.

$$(x+5)(x+7)=\overset{F}{x^2}+\overset{O}{7x}+\overset{I}{5x}+\overset{L}{35}$$

Combining like terms, $(x+5)(x+7)=x^2+12x+35$

Exercise 5.7.1

Simplify: $(x+2)(x+11)$

Answer

$x^2+13x+22$

Example 5.7.2

Use the FOIL method to simplify: $(2x-7)(x-4)$

Solution

Multiply the "First" positions: $2x^2$. Multiply the "Outer" positions: $-8x$. Multiply the "Inner" positions: $-7x$. Multiply the "Last" positions: 28.

$$(2x-7)(x-4) = \overset{F}{2x^2} - \overset{O}{8x} - \overset{I}{7x} + \overset{L}{28}$$

Combining like terms, $(2x-7)(x-4) = 2x^2-15x+28$

Exercise 5.7.2

Simplify: $(x-1)(4x+5)$

Answer

$4x^2+x-5$

At first glance, the FOIL method doesn't look like much of a shortcut. After all, if we simply use the distributive property on the product of Example $5.7.2$, we get the same quick result.

$$\begin{aligned}(2x-7)(x-4) &= 2x(x-4)-7(x-4)\\ &= 2x^2-8x-7x+28\\ &= 2x^2-15x+28\end{aligned}$$

The FOIL method becomes a true shortcut when we add the "Outer" and "Inner" results in our head.

FOIL Shortcut

To multiply two binomials, follow these steps:

1. Multiply the terms in the "First" positions.
2. Multiply the terms in the "Outer" and "Inner" positions and combine the results mentally (if they are like terms).
3. Multiply the terms in the "Last" positions.

Example 5.7.3

Use the FOIL shortcut to simplify: $(3x+8)(2x-1)$

Solution

Each of the following steps is performed mentally.

1. Multiply the terms in the "First" positions: $6x^2$
2. Multiply the terms in the "Outer" and "Inner" positions and add the results mentally: $-3x+16x=13x$
3. Multiply the terms in the "Last" positions: -8

Write the answer with no intermediate steps: $(3x+8)(2x-1) = 6x^2+13x-8$

Exercise 5.7.3

Simplify: $(2z-3)(5z-1)$

Answer

$10z^2-17z+3$

Example 5.7.4

Use the FOIL shortcut to simplify: $(4y-3)(5y+2)$

Solution

Each of the following steps is performed mentally.

1. Multiply the terms in the “First” positions: $20y^2$
2. Multiply the terms in the “Outer” and “Inner” positions and add the results mentally: $8y-15y=-7y$
3. Multiply the terms in the “Last” positions: -6

Write the answer with no intermediate steps: $(4y-3)(5y+2)=20y^2-7y-6$

Exercise 5.7.4

Simplify: $(7x+2)(2x-3)$

Answer

$14x^2-17x-6$

The Difference of Squares

We can use the FOIL shortcut to multiply $(a+b)(a-b)$.

1. Multiply the terms in the “First” positions: a^2
2. Multiply the terms in the “Outer” and “Inner” positions and add the results mentally: $ab-ab=0$
3. Multiply the terms in the “Last” positions: $-b^2$

Thus, $(a+b)(a-b)=a^2-b^2$. Note how the right-hand side a^2-b^2 is the difference of two squares. This leads to the following shortcut.

The difference of squares

If you have identical terms in the “First” positions and identical terms in the “Last” positions, but one set is separated with a plus sign while the other is separated by a minus sign, then proceed as follows:

1. Square the “First” term.
2. Square the “Last” term.
3. Place a minus sign between the results

That is,

$$(a+b)(a-b)=a^2-b^2$$

Note

If you don’t have identical terms in the “First” and “Last” positions, with one set separated with a plus sign and the other with a minus sign, then you do not have the difference of squares pattern and you must find some other way to multiply. For example, $(x+3)(x-3)$ is an example of the difference of squares pattern, but $(2y+3)(2y-5)$ is not.

Example 5.7.5

Use the difference of squares shortcut to simplify: $(x+3)(x-3)$

Solution

Note how the terms in the “First” position are identical, as are the terms in the “Last” position, with one set separated by a plus sign and the other with a minus sign. Hence, this is the difference of squares pattern and we proceed as follows:

1. Square the term in the “First” position: x^2
2. Square the term in the “Last” position: $(-3)^2=9$
3. Separate the squares with a minus sign.

That is:

$$\begin{aligned}(x+3)(x-3)&=x^2-(3)^2\\&=x^2-9\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in $(x+3)(x-3)=x^2-9$.

Exercise 5.7.5

Simplify: $(x+5)(x-5)$

Answer

x^2-25

Example 5.7.6

Use the difference of squares shortcut to simplify: $(8y+7z)(8y-7z)$

Solution

Note how the terms in the “First” position are identical, as are the terms in the “Last” position, with one set separated by a plus sign and the other with a minus sign. Hence, this is the difference of squares pattern and we proceed as follows:

1. Square the term in the “First” position: $(8y)^2=64y^2$
2. Square the term in the “Last” position: $(7z)^2=49z^2$
3. Separate the squares with a minus sign.

That is:

$$\begin{aligned}(8y+7z)(8y-7z)&=(8y)^2-(7z)^2\\&=64y^2-49z^2\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in $(8y+7z)(8y-7z)=64y^2-49z^2$

Exercise 5.7.6

Simplify: $(3a-6b)(3a+6b)$

Answer

$9a^2-36b^2$

Example 5.7.7

Use the difference of squares shortcut to simplify: $\left(x^{3}-5y^{2}\right)\left(x^{3}+5y^{2}\right)$

Solution

Note how the terms in the "First" position are identical, as are the terms in the "Last" position, with one set separated by a plus sign and the other with a minus sign. Hence, this is the difference of squares pattern and we proceed as follows:

1. Square the term in the "First" position: $(x^{3})^{2}=x^{6}$
2. Square the term in the "Last" position: $(5y^{2})^{2}=25y^{4}$
3. Separate the squares with a minus sign.

That is:

$$\begin{aligned}\left(x^{3}-5y^{2}\right)\left(x^{3}+5y^{2}\right) &= \left(x^{3}\right)^{2}-\left(5y^{2}\right)^{2} \\ &= x^{6}-25y^{4}\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in $\left(x^{3}-5y^{2}\right)\left(x^{3}+5y^{2}\right)=x^{6}-25y^{4}$

Exercise 5.7.7

Simplify: $\left(2y^{4}+z^{3}\right)\left(2y^{4}-z^{3}\right)$

Answer

$4y^{8}-z^{6}$

Squaring a Binomial

Before demonstrating the correct procedure for squaring a binomial, we will first share one of the most common mistakes made in algebra.

Warning! This is incorrect!

One of the most common mistakes made in algebra is the assumption that:

$$(a+b)^{2}=a^{2}+b^{2}$$

The fact that this is incorrect is easily checked. Substitute 3 for a and 4 for b.

$$\begin{aligned}(3+4)^{2} &= 3^{2}+4^{2} \\ 7^{2} &= 3^{2}+4^{2} \\ 49 &= 9+16\end{aligned}$$

Clearly this is incorrect!

So what is the correct answer? First, $(a+b)^{2}=(a+b)(a+b)$. We can now use the FOIL shortcut.

1. Multiply the terms in the "First" positions: a^{2}
2. Multiply the terms in the "Outer" and "Inner" positions and add the results mentally: $ab+ab=2ab$
3. Multiply the terms in the "Last" positions: b^{2}

Hence, the correct answer is $(a+b)^{2}=a^{2}+2ab+b^{2}$. This leads us to the following shortcut for squaring a binomial.

Squaring a binomial

To square a binomial, such as $(a+b)^2$, perform the following steps:

1. Square the “First” term: a^2
2. Multiply the “First” and “Last” terms and double the result: $2ab$
3. Square the “Last” term: b^2

That is:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Example 5.7.8

Use the squaring a binomial shortcut to expand: $(x+5)^2$

Solution

Follow these steps:

1. Square the first term: x^2
2. Multiply the “First” and “Last” terms and double the result: $2(x)(5) = 10x$
3. Square the “Last” term: $5^2 = 25$

Thus:

$$\begin{aligned}(x+5)^2 &= x^2 + 2(x)(5) + (5)^2 \\ &= x^2 + 10x + 25\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in $(x+5)^2 = x^2 + 10x + 25$

Exercise 5.7.8

Simplify: $(x+3)^2$

Answer

$x^2 + 6x + 9$

Comment

Students often refuse to learn the “squaring a binomial” shortcut, noting that they can just as easily use the FOIL technique or a simple application of the distributive property to arrive at the same result. Unfortunately, failure to learn the “squaring a binomial” shortcut will severely handicap students, as this pattern is an important component of many procedures in future mathematics courses.

Example 5.7.9

Use the squaring a binomial shortcut to expand: $(3x+7y)^2$

Solution

Follow these steps:

1. Square the first term: $(3x)^2 = 9x^2$
2. Multiply the “First” and “Last” terms and double the result: $2(3x)(7y) = 42xy$
3. Square the “Last” term: $(7y)^2 = 49y^2$

Thus:

$$(3x+7y)^2 = (3x)^2 + 2(3x)(7y) + (7y)^2$$

$$\begin{aligned}(3x+7y)^2 &= (3x)^2 + 2(3x)(7y) + (7y)^2 \\ &= 9x^2 + 42xy + 49y^2\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in $(3x+7y)^2 = 9x^2 + 42xy + 49y^2$

Exercise 5.7.9

Simplify: $(2y+3z)^2$

Answer

$4y^2 + 12yz + 9z^2$

In the next example, when squaring a binomial with a minus sign, we take care of the minus sign by “adding the opposite.”

Example 5.7.10

Use the squaring a binomial shortcut to expand: $\left(4a^2 - 5b^3\right)^2$

Solution

Add the opposite: $(4a^2 - 5b^3)^2 = (4a^2 + (-5b^3))^2$. Now follow these steps:

1. Square the first term: $(4a^2)^2 = 16a^4$
2. Multiply the “First” and “Last” terms and double the result: $2(4a^2)(-5b^3) = -40a^2b^3$
3. Square the “Last” term: $(-5b^3)^2 = 25b^6$

Thus:

$$\begin{aligned}\left(4a^2 - 5b^3\right)^2 &= \left(4a^2 + \left(-5b^3\right)\right)^2 \\ &= \left(4a^2\right)^2 + 2\left(4a^2\right)\left(-5b^3\right) + \left(-5b^3\right)^2 \\ &= 16a^4 - 40a^2b^3 + 25b^6\end{aligned}$$

Note

You should practice this pattern until you can go straight from the problem statement to the answer without writing down any intermediate work, as in

$$\left(4a^2 - 5b^3\right)^2 = 16a^4 - 40a^2b^3 + 25b^6$$

Exercise 5.7.10

Simplify: $\left(3x^4 - 5z^2\right)^2$

Answer

$9x^8 - 30x^4z^2 + 25z^4$

Example 5.7.10 shows us that if we are squaring a difference, the middle term will be minus. That is, the only difference between $(a+b)^2$ and $(a-b)^2$ is the sign of the middle term.

Squaring a binomial

The shortcuts for squaring a binomial are:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Example 5.7.11

Use the squaring a binomial shortcut to expand: $\left(x^3-y^3\right)^2$

Solution

Use the pattern $(a-b)^2=a^2-2ab+b^2$. Square the “First” term, multiply the “First” and “Last” terms and double the result, then square the “Last” term. Because of the minus sign, the middle term will be minus, but all other terms are plus.

$$\begin{aligned}\left(x^3-y^3\right)^2 &= \left(x^3\right)^2-2\left(x^3\right)\left(y^3\right)+\left(y^3\right)^2\\ &= x^6-2x^3y^3+y^6\end{aligned}$$

Exercise 5.7.11

Simplify: $\left(a^2-3b^5\right)^2$

Answer

$a^4-6a^2b^5+9b^{10}$

An Application

In Example 5.7.7, we found the area of the outer square by summing the areas of its parts (see Figure 5.7.1). Recall that the answer was $A=x^2+6x+9$.

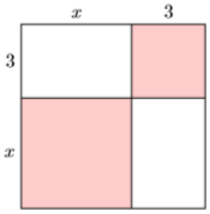

Figure 5.7.1: Find the area of the square.

Now that we have the squaring a binomial shortcut, we can simplify the process of finding the area of the outer square by squaring its side. That is:

$$A=(x+3)^2$$

Now we can use the squaring the binomial technique to expand.

$$\begin{aligned}&= x^2+2(x)(3)+(3)^2\\ &= x^2+6x+9\end{aligned}$$

Note that this is the same as the answer found by summing the four parts of the square in Example 5.7.7.

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5.E: Polynomial Functions (Exercises)

5.1: Functions

In Exercises 1-6, state the domain and range of the given relation.

1) $R=\{(7,4),(2,4),(4,2),(8,5)\}$

Answer

Domain $=\{2,4,7,8\}$ and Range $=\{2,4,5\}$

2) $S=\{(6,4),(3,3),(2,5),(8,7)\}$

3) $T=\{(7,2),(3,1),(9,4),(8,1)\}$

Answer

Domain $=\{3,7,8,9\}$ and Range $=\{1,2,4\}$

4) $R=\{(0,1),(8,2),(6,8),(9,3)\}$

5) $T=\{(4,7),(4,8),(5,0),(0,7)\}$

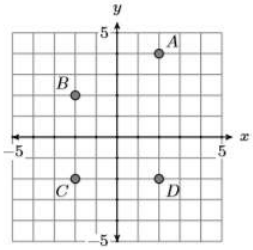
Answer

Domain $=\{0,4,5\}$ and Range $=\{0,7,8\}$

6) $T=\{(9,0),(3,6),(8,0),(3,8)\}$

In Exercises 7-10, state the domain and range of the given relation.

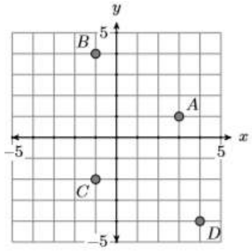
7)



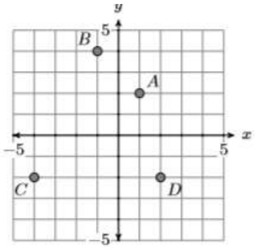
Answer

Domain $=\{-2,2\}$ and Range $=\{-2,2,4\}$

8)



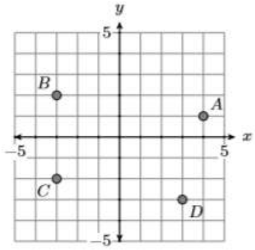
9)



Answer

Domain $= \{-4, -1, 1, 2\}$ and Range $= \{-2, 2, 4\}$

10)



In Exercises 11-18, determine whether the given relation is a function.

11) $R = \{(-6, -4), (-4, -4), (1, -4)\}$

Answer

Function

12) $T = \{(-8, -3), (-4, -3), (2, -3)\}$

13) $T = \{(-1, -7), (2, -5), (4, -2)\}$

Answer

Function

14) $S = \{(-6, -6), (-4, 0), (9, 1)\}$

15) $T = \{(-9, 1), (1, 6), (1, 8)\}$

Answer

Not a function

16) $S = \{(-7, 0), (1, 1), (1, 2)\}$

17) $R = \{(-7, -8), (-7, -6), (-5, 0)\}$

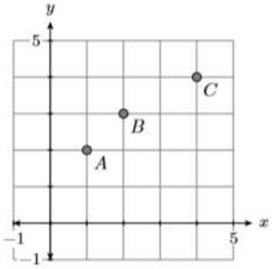
Answer

Not a function

18) $T = \{(-8, -9), (-8, -4), (-5, 9)\}$

In Exercises 19-22, determine whether the given relation is a function.

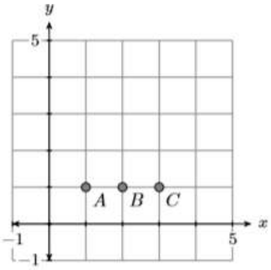
19)



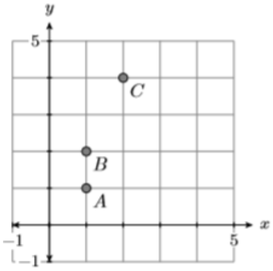
Answer

Function

20)



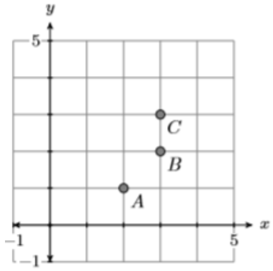
21)



Answer

Not a function

22)



23) Given $f(x) = |6x - 9|$, evaluate $f(8)$.

Answer

39

24) Given $f(x)=|8x-3|$, evaluate $f(5)$.

25) Given $f(x)=-2x^2+8$, evaluate $f(3)$.

Answer

-10

26) Given $f(x)=3x^2+x+6$, evaluate $f(-3)$.

27) Given $f(x)=-3x^2+4x+1$, evaluate $f(2)$.

Answer

-3

28) Given $f(x)=-3x^2+4x-2$, evaluate $f(2)$.

29) Given $f(x)=|5x+9|$, evaluate $f(-8)$.

Answer

31

30) Given $f(x)=|9x-6|$, evaluate $f(4)$.

31) Given $f(x)=\sqrt{x-6}$, evaluate $f(42)$.

Answer

6

32) Given $f(x)=\sqrt{x+8}$, evaluate $f(41)$.

33) Given $f(x)=\sqrt{x-7}$, evaluate $f(88)$.

Answer

9

34) Given $f(x)=\sqrt{x+9}$, evaluate $f(16)$.

35) Given $f(x)=-4x+6$, evaluate $f(8)$.

Answer

-26

36) Given $f(x)=-9x+2$, evaluate $f(-6)$.

37) Given $f(x)=-6x+7$, evaluate $f(8)$.

Answer

-41

38) Given $f(x)=-6x-2$, evaluate $f(5)$.

39) Given $f(x)=-2x^2+3x+2$ and $g(x)=3x^2+5x-5$, evaluate $f(3)$ and $g(3)$.

Answer

$f(3)=-7$ and $g(3)=37$

40) Given $f(x)=3x^2-3x-5$ and $g(x)=2x^2-5x-8$, evaluate $f(-2)$ and $g(-2)$.

41) Given $f(x)=6x-2$ and $g(x)=-8x+9$, evaluate $f(-7)$ and $g(-7)$.

Answer

$f(-7)=-44$ and $g(-7)=65$

42) Given $f(x)=5x-3$ and $g(x)=9x-9$, evaluate $f(-2)$ and $g(-2)$.

43) Given $f(x)=4x-3$ and $g(x)=-3x+8$, evaluate $f(-3)$ and $g(-3)$.

Answer

$f(-3)=-15$ and $g(-3)=17$

44) Given $f(x)=8x+7$ and $g(x)=2x-7$, evaluate $f(-9)$ and $g(-9)$.

45) Given $f(x)=-2x^2+5x-9$ and $g(x)=-2x^2+3x-4$, evaluate $f(-2)$ and $g(-2)$.

Answer

$f(-2)=-27$ and $g(-2)=-18$

46) Given $f(x)=-3x^2+5x-2$ and $g(x)=3x^2-4x+2$, evaluate $f(-1)$ and $g(-1)$.

5.2: Polynomials

In Exercises 1-6, state the coefficient and the degree of each of the following terms.

1) $3v^5u^6$

Answer

Coefficient $=3$, Degree $=11$

2) $-3b^5z^8$

3) $-5v^6$

Answer

Coefficient $=-5$, Degree $=6$

4) $-5c^3$

5) $2u^7x^4d^5$

Answer

Coefficient $=2$, Degree $=16$

6) $9w^4c^5u^7$

In Exercises 7-16, state whether each of the following expressions is a monomial, binomial, or trinomial.

7) $-7b^9c^3$

Answer

Monomial

8) $7b^6c^2$

9) $4u+7v$

Answer

Binomial

10) $-3b+5c$

11) $3b^4 - 9bc + 9c^2$

Answer

Trinomial

12) $8u^4 + 5uv + 3v^4$

13) $5s^2 + 9t^7$

Answer

Binomial

14) $-8x^6 - 6y^7$

15) $2u^3 - 5uv - 4v^4$

Answer

Trinomial

16) $6y^3 - 4yz + 7z^3$

In Exercises 17-20, sort each of the given polynomials in descending powers of x.

17) $-2x^7 - 9x^{13} - 6x^{12} - 7x^{17}$

Answer

$-7x^{17} - 9x^{13} - 6x^{12} - 2x^7$

18) $2x^4 - 8x^{19} + 3x^{10} - 4x^2$

19) $8x^6 + 2x^{15} - 3x^{11} - 2x^2$

Answer

$2x^{15} - 3x^{11} + 8x^6 - 2x^2$

20) $2x^6 - 6x^7 - 7x^{15} - 9x^{18}$

In Exercises 21-24, sort each of the given polynomials in ascending powers of x.

21) $7x^{17} + 3x^4 - 2x^{12} + 8x^{14}$

Answer

$3x^4 - 2x^{12} + 8x^{14} + 7x^{17}$

22) $6x^{18} - 6x^4 - 2x^{19} - 7x^{14}$

23) $2x^{13} + 3x^{18} + 8x^7 + 5x^4$

Answer

$5x^4 + 8x^7 + 2x^{13} + 3x^{18}$

24) $-6x^{18} - 8x^{11} - 9x^{15} + 5x^{12}$

In Exercises 25-32, simplify the given polynomial, combining like terms, then arranging your answer in descending powers of x.

25) $-5x + 3 - 6x^3 + 5x^2 - 9x + 3 - 3x^2 + 6x^3$

Answer

$2x^2 - 14x + 6$

26) $-2x^3+8x-x^2+5+7+6x^2+4x^3-9x$

27) $4x^3+6x^2-8x+1+8x^3-7x^2+5x-8$

Answer

$12x^3-x^2-3x-7$

28) $-8x^3-2x^2-7x-3+7x^3-9x^2-8x+9$

29) $x^2+9x-3+7x^2-3x-8$

Answer

$8x^2+6x-11$

30) $-4x^2-6x+3-3x^2+3x-6$

31) $8x+7+2x^2-8x-3x^3-x^2$

Answer

$-3x^3+x^2+7$

32) $-x^2+8-7x+8x-5x^2+4x^3$

In Exercises 33-44, simplify the given polynomial, combining like terms, then arranging your answer in a reasonable order, perhaps in descending powers of either variable. ***Note: Answers may vary, depending on which variable you choose to dictate the order.***

33) $-8x^2-4xz-2z^2-3x^2-8xz+2z^2$

Answer

$-11x^2-12xz$

34) $-5x^2+9xz-4z^2-6x^2-7xz+7z^2$

35) $-6u^3+4uv^2-2v^3-u^3+6u^2v-5uv^2$

Answer

$-7u^3+6u^2v-uv^2-2v^3$

36) $7a^3+6a^2b-5ab^2+4a^3+6a^2b+6b^3$

37) $-4b^2c-3bc^2-5c^3+9b^3-3b^2c+5bc^2$

Answer

$9b^3-7b^2c+2bc^2-5c^3$

38) $4b^3-6b^2c+9bc^2-9b^3-8bc^2+3c^3$

39) $-8y^2+6yz-7z^2-2y^2-3yz-9z^2$

Answer

$-10y^2+3yz-16z^2$

40) $8x^2+xy+3y^2-x^2+7xy+y^2$

41) $7b^2c+8bc^2-6c^3-4b^3+9bc^2-6c^3$

Answer

$-4b^3+7b^2c+17bc^2-12c^3$

42) $7x^3 - 9x^2y + 3y^3 + 7x^3 + 3xy^2 - 7y^3$

43) $9a^2 + ac - 9c^2 - 5a^2 - 2ac + 2c^2$

Answer

$4a^2 - ac - 7c^2$

44) $7u^2 + 3uv - 6v^2 - 6u^2 + 7uv + 6v^2$

In Exercises 45-50, state the degree of the given polynomial.

45) $3x^{15} + 4 + 8x^3 - 8x^{19}$

Answer

19

46) $-4x^6 - 7x^{16} - 5 + 3x^{18}$

47) $7x^{10} - 3x^{18} + 9x^4 - 6$

Answer

18

48) $3x^{16} - 8x^5 + x^8 + 7$

49) $-2 - x^7 - 5x^5 + x^{10}$

Answer

10

50) $x^{11} + 7x^{16} + 8 - 7x^{10}$

51) Given $f(x) = 5x^3 + 4x^2 - 6$, evaluate $f(-1)$.

Answer

-7

52) Given $f(x) = -3x^3 + 3x^2 - 9$, evaluate $f(-1)$.

53) Given $f(x) = 5x^4 - 4x - 6$, evaluate $f(-2)$.

Answer

82

54) Given $f(x) = -2x^4 - 4x - 9$, evaluate $f(2)$.

55) Given $f(x) = 3x^4 + 5x^3 - 9$, evaluate $f(-2)$.

Answer

-1

56) Given $f(x) = -3x^4 + 2x^3 - 6$, evaluate $f(-1)$.

57) Given $f(x) = 3x^4 - 5x^2 + 8$, evaluate $f(-1)$.

Answer

6

58) Given $f(x) = -4x^4 - 5x^2 - 3$, evaluate $f(3)$.

59) Given $f(x) = -2x^3 + 4x - 9$, evaluate $f(2)$.

Answer

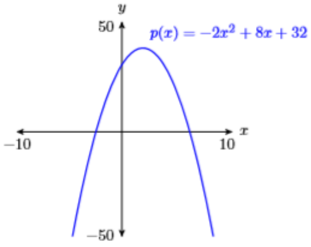
-17

60) Given $f(x) = 4x^3 + 3x + 7$, evaluate $f(-2)$.

In Exercises 61-64, use your graphing calculator to sketch the the given quadratic polynomial. In each case the graph is a parabola, so adjust the WINDOW parameters until the vertex is visible in the viewing window, then follow the Calculator Submission Guidelines when reporting your solution on your homework.

61) $p(x) = -2x^2 + 8x + 32$

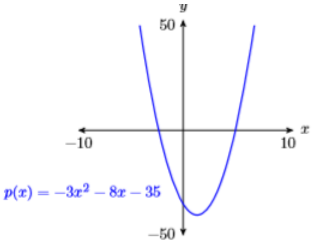
Answer



62) $p(x) = 2x^2 + 6x - 18$

63) $p(x) = 3x^2 - 8x - 35$

Answer

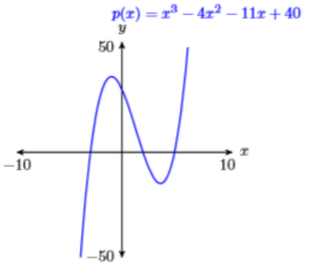


64) $p(x) = -4x^2 - 9x + 50$

In Exercises 65-68, use your graphing calculator to sketch the polynomial using the given WINDOW parameters. Follow the Calculator Submission Guidelines when reporting your solution on your homework.

65) $p(x) = x^3 - 4x^2 - 11x + 30$
$\mathbf{X}\min = -10 \quad \mathbf{X}\max = 10$
$\mathbf{Y}\min = -50 \quad \mathbf{Y}\max = 50$

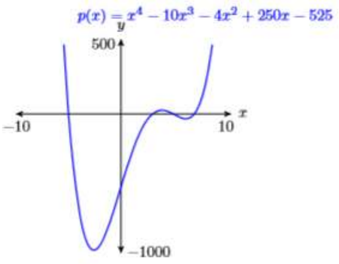
Answer



66) $p(x)=-x^3+4x^2+27x-90$
$\mathbf{X}\min=-10 \quad \mathbf{X}\max=10$
$\mathbf{Y}\min=-150 \quad \mathbf{Y}\max=50$

67) $p(x)=x^4-10x^3-4x^2+250x-525$
$\mathbf{X}\min=-10 \quad \mathbf{X}\max=10$
$\mathbf{Y}\min=-1000 \quad \mathbf{Y}\max=500$

Answer



68) $p(x)=-x^4+2x^3+35x^2-36x-180$
$\mathbf{X}\min=-10 \quad \mathbf{X}\max=10$
$\mathbf{Y}\min=-50 \quad \mathbf{Y}\max=50$

5.3: Applications of Polynomials

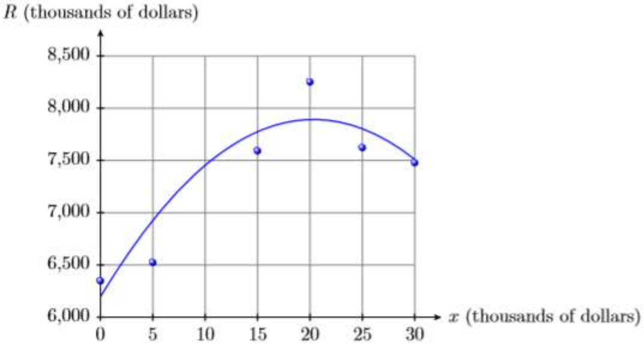
1) A firm collects data on the amount it spends on advertising and the resulting revenue collected by the firm. Both pieces of data are in thousands of dollars.

x (advertising costs)	0	5	15	20	25	30
R (revenue)	6347	6524	7591	8251	7623	7478

The data is plotted then fitted with the following second degree polynomial, where x is the amount invested in thousands of dollars and $R(x)$ is the amount of revenue earned by the firm (also in thousands of dollars).

$$R(x)=-4.1x^2+166.8x+6196$$

Use the graph and then the polynomial to estimate the firm's revenue when the firm invested $\$10,000$ in advertising.



Answer

Approximately $\$7,454,000$

2) The table below lists the estimated number of aids cases in the United States for the years 1999-2003.

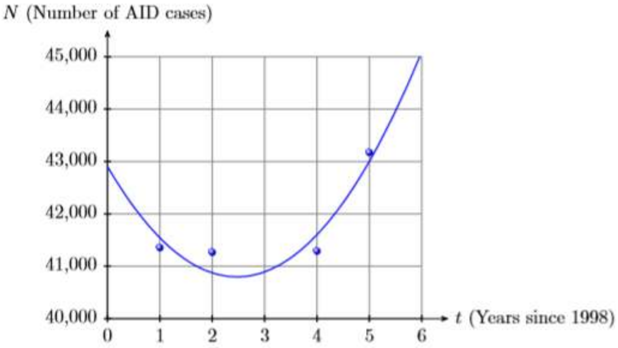
Year	1999	2000	2002	2003

AIDS Cases	41,356	41,267	41,289	43,171

The data is plotted then fitted with the following second degree polynomial, where t is the number of years that have passed since 1998 and $N(t)$ is the number of aids case reported t years after 1998.

$$N(t) = 345.14t^2 - 1705.7t + 42904$$

Use the graph and then the polynomial to estimate the number of AIDS cases in the year 2001.



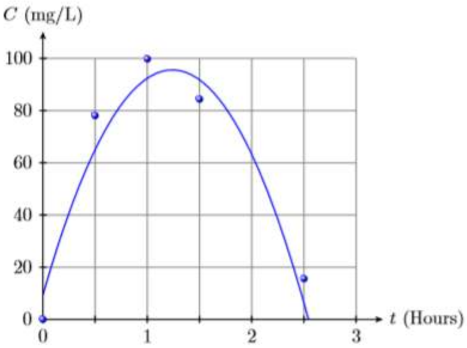
3) The following table records the concentration (in milligrams per liter) of medication in a patient's blood after indicated times have passed.

Time (Hours)	0	0.5	1	0.5	2.5
Concentration(mg/L)	0	78.1	99.8	84.4	15.6

The data is plotted then fitted with the following second degree polynomial, where t is the number of hours that have passed since taking the medication and $C(t)$ is the concentration (in milligrams per liter) of the medication in the patient's blood after t hours have passed.

$$C(t) = -56.214t^2 + 139.31t + 9.35$$

Use the graph and then the polynomial to estimate the the concentration of medication in the patient's blood 2 hours after taking the medication.



Answer

Approximately 63mg/L

4) The following table records the population (in millions of people) of the United States for the given year.

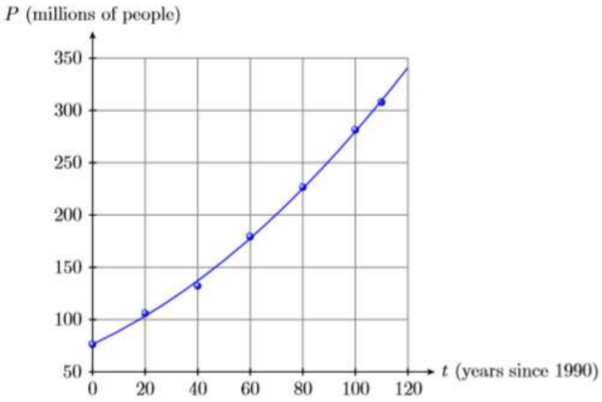
Year	1900	1920	1940	1960	1980	2000	2010

Population (millions)	76.2	106.0	132.2	179.3	226.5	281.4	307.7

The data is plotted then fitted with the following second degree polynomial, where t is the number of years that have passed since 1990 and $P(t)$ is the population (in millions) t years after 1990.

$P(t) = 0.008597t^2 + 1,1738t + 76.41$

Use the graph and then the polynomial to estimate the the population of the United States in the year 1970.



5) If a projectile is launched with an initial velocity of 457 meters per second ($457\mathrm{m/s}$) from a rooftop 75 meters ($75\mathrm{m}$) above ground level, at what time will the projectile first reach a height of 6592 meters ($6592\mathrm{m}$)? Round your answer to the nearest second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second (*$9.8\mathrm{m/s}^2$*).*

Answer

17.6 seconds

6) If a projectile is launched with an initial velocity of 236 meters per second ($236\mathrm{m/s}$) from a rooftop 15 meters ($15\mathrm{m}$) above ground level, at what time will the projectile first reach a height of 1838 meters ($1838\mathrm{m}$)? Round your answer to the nearest second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second (*$9.8\mathrm{m/s}^2$*).*

7) If a projectile is launched with an initial velocity of 229 meters per second ($229\mathrm{m/s}$) from a rooftop 58 meters ($58\mathrm{m}$) above ground level, at what time will the projectile first reach a height of 1374 meters ($1374\mathrm{m}$)? Round your answer to the nearest second.

Note: The acceleration due to gravity near the earth's surface is 9.8 meters per second per second ($9.8\mathrm{m/s}^2$*).*

Answer

6.7 seconds

8) If a projectile is launched with an initial velocity of 234 meters per second ($234\mathrm{m/s}$) from a rooftop 16 meters ($16\mathrm{m}$) above ground level, at what time will the projectile first reach a height of 1882 meters ($1882\mathrm{m}$)? Round your answer to the nearest second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second (*$9.8\mathrm{m/s}^2$*).*

In Exercises 9-12, first use an algebraic technique to find the zero of the given function, then use the 2:zero utility on your graphing calculator to locate the zero of the function. Use the Calculator Submission Guidelines when reporting the zero found using your graphing calculator.

9) $f(x) = 3.25x - 4.875$

Answer

Zero: 1.5

10) $f(x) = 3.125 - 2.5x$

11) $f(x) = 3.9 - 1.5x$

Answer

Zero: 2.6

12) $f(x) = 0.75x + 2.4$

13) If a projectile is launched with an initial velocity of 203 meters per second $(203\mathrm{m/s})$ from a rooftop 52 meters $(52\mathrm{m})$ above ground level, at what time will the projectile return to ground level? Round your answer to the nearest tenth of a second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second* $(9.8\mathrm{m/s^2})$.

Answer

41.7 seconds

14) If a projectile is launched with an initial velocity of 484meters per second $(484\mathrm{m/s})$ from a rooftop 17 meters $(17\mathrm{m})$ above ground level, at what time will the projectile return to ground level? Round your answer to the nearest tenth of a second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second* $(9.8\mathrm{m/s^2})$.

15) If a projectile is launched with an initial velocity of 276 meters per second $(276\mathrm{m/s})$ from a rooftop 52 meters $(52\mathrm{m})$ above ground level, at what time will the projectile return to ground level? Round your answer to the nearest tenth of a second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second* $(9.8\mathrm{m/s^2})$.

Answer

56.5 seconds

16) If a projectile is launched with an initial velocity of 204 meters per second $(204\mathrm{m/s})$ from a rooftop 92 meters $(92\mathrm{m})$ above ground level, at what time will the projectile return to ground level? Round your answer to the nearest tenth of a second.

Note: The acceleration due to gravity near the earth's surface is 9.8 *meters per second per second* $(9.8\mathrm{m/s^2})$.

5.4: Adding and Subtracting Polynomials

In Exercises 1-8, simplify the given expression. Arrange your answer in some sort of reasonable order.

1) $(-8r^2t + 7rt^2 + 3t^3) + (9r^3 + 2rt^2 + 4t^3)$

Answer

$9r^3 - 8r^2t + 9rt^2 + 7t^3$

2) $(-a^3 - 8ac^2 - 7c^3) + (-7a^3 - 8a^2c + 8ac^2)$

3) $(7x^2 - 6x - 9) + (8x^2 + 10x + 9)$

Answer

$15x^2 + 4x$

4) $(-7x^2 + 5x - 6) + (-10x^2 - 1)$

5) $(-2r^2 + 7rs + 4s^2) + (-9r^2 + 7rs - 2s^2)$

Answer

$-11r^2 + 14rs + 2s^2$

6) $(-2r^2 + 3rt - 4t^2) + (7r^2 + 4rt - 7t^2)$

7) $(-8y^3 - 3y^2z - 6z^3) + (-3y^3 + 7y^2z - 9yz^2)$

Answer

$-11y^3+4y^2z-9yz^2-6z^3$

8) $(7y^2z+8yz^2+2z^3)+(8y^3-8y^2z+9yz^2)$

In Exercises 9-14, simplify the given expression by distributing the minus sign.

9) $-(5x^2-4)$

Answer

$-5x^2+4$

10) $-(-8x^2-5)$

11) $-(9r^3-4r^2t-3rt^2+4t^3)$

Answer

$-9r^3+4r^2t+3rt^2-4t^3$

12) $-(7u^3-8u^2v+6uv^2+5v^3)$

13) $-(-5x^2+9xy+6y^2)$

Answer

$5x^2-9xy-6y^2$

14) $-(-4u^2-6uv+5v^2)$

In Exercises 15-22, simplify the given expression. Arrange your answer in some sort of reasonable order.

15) $(-u^3-4u^2w+7w^3)-(u^2w+uw^2+3w^3)$

Answer

$-u^3-5u^2w-uw^2+4w^3$

16) $(-b^2c+8bc^2+8c^3)-(6b^3+b^2c-4bc^2)$

17) $(2y^3-2y^2z+3z^3)-(-8y^3+5yz^2-3z^3)$

Answer

$10y^3-2y^2z-5yz^2+6z^3$

18) $(4a^3+6ac^2+5c^3)-(2a^3+8a^2c-7ac^2)$

19) $(-7r^2-9rs-2s^2)-(-8r^2-7rs+9s^2)$

Answer

$r^2-2rs-11s^2$

20) $(-4a^2+5ab-2b^2)-(-8a^2+7ab+2b^2)$

21) $(10x^2+2x-6)-(-8x^2+14x+17)$

Answer

$18x^2-12x-23$

22) $(-5x^2+19x-5)-(-15x^2+19x+8)$

In Exercises 23-28, for the given polynomial functions $f(x)$ and $g(x)$, simplify $f(x)+g(x)$. Arrange your answer in descending powers of x.

23) $f(x) = -2x^2 + 9x + 7$
$g(x) = 8x^3 - 7x^2 + 5$

Answer

$8x^3 - 9x^2 + 9x + 12$

24) $f(x) = -8x^3 + 6x - 9$
$g(x) = x^3 - x^2 + 3x$

25) $f(x) = 5x^3 - 5x^2 + 8x$
$g(x) = 7x^2 - 2x - 9$

Answer

$5x^3 + 2x^2 + 6x - 9$

26) $f(x) = -x^2 + 8x + 1$
$g(x) = -7x^3 + 8x - 9$

27) $f(x) = -3x^2 - 8x - 9$
$g(x) = 5x^2 - 4x + 4$

Answer

$2x^2 - 12x - 5$

28) $f(x) = -3x^2 + x - 8$
$g(x) = 7x^2 - 9$

In Exercises 29-34, for the given polynomial functions $f(x)$ and $g(x)$, simplify $f(x) - g(x)$. Arrange your answer in descending powers of x.

29) $f(x) = -6x^3 - 7x + 7$
$g(x) = -3x^3 - 3x^2 - 8x$

Answer

$-3x^3 + 3x^2 + x + 7$

30) $f(x) = 5x^3 - 5x + 4$
$g(x) = -8x^3 - 2x^2 - 3x$

31) $f(x) = 12x^2 - 5x + 4$
$g(x) = 8x^2 - 16x - 7$

Answer

$4x^2 + 11x + 11$

32) $f(x) = -7x^2 + 12x + 17$
$g(x) = -10x^2 - 17$

33) $f(x) = -3x^3 - 4x + 2$
$g(x) = -4x^3 - 7x^2 + 6$

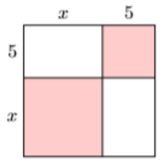
Answer

$x^3 + 7x^2 - 4x - 4$

34) $f(x) = -9x^2 + 9x + 3$
$g(x) = 7x^3 + 7x^2 + 5$

In Exercises 35-36, find the area of the given square by summing the areas of its four parts.

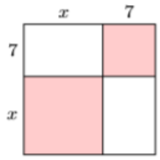
35)



Answer

$x^2+10x+25$

36)



37) Rachel runs a small business selling wicker baskets. Her business costs for producing and selling x wicker baskets are given by the polynomial function $C(x) = 232 + 7x - 0.0085x^2$. The revenue she earns from selling x wicker baskets is given by the polynomial function $R(x) = 33.45x$. Find a formula for $P(x)$, the profit made from selling x wicker baskets. Use your formula to determine Rachel's profit if she sells 233 wicker baskets. Round your answer to the nearest cent.

Answer

$\$6,392.31$

38) Eloise runs a small business selling baby cribs. Her business costs for producing and selling x baby cribs are given by the polynomial function $C(x) = 122 + 8x - 0.0055x^2$. The revenue she earns from selling x baby cribs is given by the polynomial function $R(x) = 33.45x$. Find a formula for $P(x)$, the profit made from selling x baby cribs. Use your formula to determine Eloise's profit if she sells 182 baby cribs. Round your answer to the nearest cent.

5.5: Laws of Exponents

In Exercises 1-8, simplify each of the given exponential expressions.

1) $(-4)^3$

Answer

-64

2) $(-9)^2$

3) $\left(-\frac{5}{7}\right)^0$

Answer

1

4) $\left(-\frac{2}{5}\right)^0$

5) $\left(-\frac{4}{3}\right)^2$

Answer

$\frac{16}{9}$

6) $\left(-\frac{2}{3}\right)^2$

7) $(-19)^0$

Answer

1

8) $(-17)^0$

In Exercises 9-18, simplify each of the given exponential expressions.

9) $(7v-6w)^{18}\cdot(7v-6w)^{17}$

Answer

$(7v-6w)^{35}$

10) $(8a+7c)^3\cdot(8a+7c)^{19}$

11) $3^4\cdot 3^0$

Answer

3^4

12) $5^7\cdot 5^0$

13) $4^n\cdot 4^{8n+3}$

Answer

4^{9n+3}

14) $4^{6m+5}\cdot 4^{m-5}$

15) $x^8\cdot x^3$

Answer

x^{11}

16) $a^9\cdot a^{15}$

17) $2^5\cdot 2^3$

Answer

2^8

18) $2^{10}\cdot 2^3$

In Exercises 19-28, simplify each of the given exponential expressions.

19) $\frac{4^{16}}{4^{16}}$

Answer

1

20) $\frac{3^{12}}{3^{12}}$

21) $\frac{w^{11}}{w^7}$

Answer

w^4

22) $\frac{c^{10}}{c^8}$

23) $\frac{(9a-8c)^{15}}{(9a-8c)^8}$

Answer

$(9a-8c)^7$

24) $\frac{(4b+7c)^{15}}{(4b+7c)^5}$

25) $\frac{2^{9n+5}}{2^{3n-4}}$

Answer

2^{6n+9}

26) $\frac{2^{4k-9}}{2^{3k-8}}$

27) $\frac{4^{17}}{4^9}$

Answer

4^8

28) $\frac{2^{17}}{2^6}$

In Exercises 29-38, simplify each of the given exponential expressions.

29) $(4^{8m-6})^7$

Answer

4^{56m-42}

30) $(2^{2m-9})^3$

31) $[(9x+5y)^3]^7$

Answer

$(9x+5y)^{21}$

32) $[(4u-v)^8]^9$

33) $(4^3)^2$

Answer

4^6

34) $(3^4)^2$

35) $(c^4)^7$

Answer

c^{28}

36) $(w^9)^5$

37) $(6^2)^0$

Answer

1

38) $(8^9)^0$

In Exercises 39-48, simplify each of the given exponential expressions.

39) $(uw)^5$

Answer

u^5w^5

40) $(ac)^4$

41) $(-2y)^3$

Answer

$-8y^3$

42) $(-2b)^3$

43) $(3w^9)^4$

Answer

$81w^{36}$

44) $(-3u^9)^4$

45) $(-3x^8y^2)^4$

Answer

$81x^{32}y^8$

46) $(2x^8z^6)^4$

47) $(7s^{6n})^3$

Answer

$343s^{18n}$

48) $(9b^{6n})^3$

In Exercises 49-56, simplify each of the given exponential expressions.

49) $\left(\frac{v}{2}\right)^3$

Answer

$\frac{v^3}{8}$

50) $\left(\frac{t}{9}\right)^2$

51) $\left(-\frac{2}{u}\right)^2$

Answer

$\frac{4}{u^2}$

52) $\left(-\frac{3}{w}\right)^3$

53) $\left(-\frac{r^8}{5}\right)^4$

Answer

$\frac{r^{32}}{625}$

54) $\left(-\frac{x^{11}}{5}\right)^5$

55) $\left(\frac{5}{c^9}\right)^4$

Answer

$\frac{625}{c^{36}}$

56) $\left(\frac{5}{u^{12}}\right)^2$

57) Complete each of the laws of exponents presented in the first column, then use the results to simplify the expressions in the second column.

$a^m a^n = ?$	$a^3 a^5 = ?$
$\frac{a^m}{a^n} = ?$	$\frac{a^6}{a^2} = ?$
$(a^m)^n = ?$	$(a^5)^7 = ?$
$(ab)^m = ?$	$(ab)^9 = ?$
$\left(\frac{a}{b}\right)^m = ?$	$\left(\frac{a}{b}\right)^3 = ?$

Answer

The general answers are: $a^{m+n}, a^{m-n}, a^{mn}, a^m b^m, \frac{a^m}{b^m}$.

The specific answers are: $a^8, a^4, a^{35}, a^9 b^9, \frac{a^3}{b^3}$.

5.6: Multiplying Polynomials

In Exercises 1-10, simplify the given expression.

1) $-3(7r)$

Answer

$-21r$

2) $7(3a)$

3) $(-9b^3)(-8b^6)$

Answer

$72b^9$

4) $(8s^3)(-7s^4)$

5) $(-7r^2t^4)(7r^5t^2)$

Answer

$-49r^7t^6$

6) $\left(-10 s^{2} t^{8}\right)\left(-7 s^{4} t^{3}\right)$

7) $\left(-5 b^{2} c^{9}\right)\left(-8 b^{4} c^{4}\right)$

Answer

$40 b^{6} c^{13}$

8) $\left(-9 s^{2} t^{8}\right)\left(7 s^{5} t^{4}\right)$

9) $\left(-8 v^{3}\right)\left(4 v^{4}\right)$

Answer

$-32 v^{7}$

10) $\left(-9 y^{3}\right)\left(3 y^{5}\right)$

In Exercises 11-22, use the distributive property to expand the given expression.

11) $9\left(-2 b^{2}+2 b+9\right)$

Answer

$-18 b^{2}+18 b+81$

12) $9\left(-4 b^{2}+7 b-8\right)$

13) $-4\left(10 t^{2}-7 t-6\right)$

Answer

$-40 t^{2}+28 t+24$

14) $-5\left(-7 u^{2}-7 u+2\right)$

15) $-8 u^{2}\left(-7 u^{3}-8 u^{2}-2 u+10\right)$

Answer

$56 u^{5}+64 u^{4}+16 u^{3}-80 u^{2}$

16) $-3 s^{2}\left(-7 s^{3}-9 s^{2}+6 s+3\right)$

17) $10 s^{2}\left(-10 s^{3}+2 s^{2}+2 s+8\right)$

Answer

$-100 s^{5}+20 s^{4}+20 s^{3}+80 s^{2}$

18) $8 u^{2}\left(9 u^{3}-5 u^{2}-2 u+5\right)$

19) $2 s t\left(-4 s^{2}+8 s t-10 t^{2}\right)$

Answer

$-8 s^{3} t+16 s^{2} t^{2}-20 s t^{3}$

20) $7 u v\left(-9 u^{2}-3 u v+4 v^{2}\right)$

21) $-2 u w\left(10 u^{2}-7 u w-2 w^{2}\right)$

Answer

$-20 u^{3} w+14 u^{2} w^{2}+4 u w^{3}$

22) $-6 v w\left(-5 v^{2}+9 v w+5 w^{2}\right)$

In Exercises 23-30, use the technique demonstrated in Example 5.6.8 and Example 5.6.9 to expand each of the following expressions using the distributive property.

23) $(-9x-4)(-3x+2)$

Answer

$27x^2-6x-8$

24) $(4x-10)(-2x-6)$

25) $(3x+8)(3x-2)$

Answer

$9x^2+18x-16$

26) $(-6x+8)(-x+1)$

27) $-12x^3+14x^2+6x-5$

Answer

$-\frac{930}{289}$

28) $(4x-6)\left(-7x^2-10x+10\right)$

29) $(x-6)\left(-2x^2-4x-4\right)$

Answer

$-2x^3+8x^2+20x+24$

30) $(5x-10)\left(-3x^2+7x-8\right)$

In Exercises 31-50, use the shortcut technique demonstrated in Example 5.6.10, Example 5.6.11, and Example 5.6.12 to expand each of the following expressions using the distributive property.

31) $(8u-9w)(8u-9w)$

Answer

$64u^2-144uw+81w^2$

32) $(3b+4c)(-8b+10c)$

33) $(9r-7t)(3r-9t)$

Answer

$27r^2-102rt+63t^2$

34) $(-6x-3y)(-6x+9y)$

35) $(4r-10s)\left(-10r^2+10rs-7s^2\right)$

Answer

$-40r^3+140r^2s-128rs^2+70s^3$

36) $(5s-9t)\left(-3s^2+4st-9t^2\right)$

37) $(9x-2z)\left(4x^2-4xz-10z^2\right)$

Answer

$36x^3-44x^2z-82xz^2+20z^3$

38) $\left(r-4t\right)\left(7r^2+4rt-2t^2\right)$

39) $(9r+3t)^2$

Answer

$81r^2+54rt+9t^2$

40) $(4x+8z)^2$

41) $(4y+5z)(4y-5z)$

Answer

$16y^2-25z^2$

42) $(7v+2w)(7v-2w)$

43) $(7u+8v)(7u-8v)$

Answer

$49u^2-64v^2$

44) $(6b+8c)(6b-8c)$

45) $(7b+8c)^2$

Answer

$49b^2+112bc+64c^2$

46) $(2b+9c)^2$

47) $\left(2t^2+9t+4\right)\left(2t^2+9t+4\right)$

Answer

$4t^4+36t^3+97t^2+72t+16$

48) $\left(3a^2-9a+4\right)\left(3a^2-9a+2\right)$

49) $\left(4w^2+3w+5\right)\left(3w^2-6w+8\right)$

Answer

$12w^4-15w^3+29w^2-6w+40$

50) $\left(4s^2+3s+8\right)\left(2s^2+4s-9\right)$

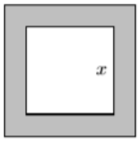
51) The demand for widgets is given by the function $x=320-0.95p$, where x is the demand and p is the unit price. What unit price should a retailer charge for widgets in order that his revenue from sales equals $\$7,804$? Round your answers to the nearest cent.

Answer

$\$26.47$, $\$310.37$

52) The demand for widgets is given by the function $x=289-0.91p$, where x is the demand and p is the unit price. What unit price should a retailer charge for widgets in order that his revenue from sales equals $\$7,257$? Round your answers to the nearest cent.

53) In the image that follows, the edge of the outer square is 6 inches longer than 3 times the edge of the inner square.

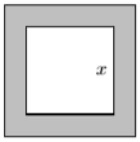


a. Express the area of the shaded region as a polynomial in terms of x, the edge of the inner square. Your final answer must be presented as a second degree polynomial in the form $A(x) = ax^2 + bx + c$.
b. Given that the edge of the inner square is 5 inches, use the polynomial in part (a) to determine the area of the shaded region.

Answer

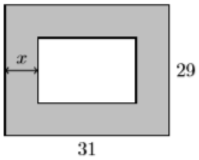
$A(x) = 8x^2 + 36x + 36$, $A(5) = 416$ square inches

54) In the image that follows, the edge of the outer square is 3 inches longer than 2 times the edge of the inner square.



a. Express the area of the shaded region as a polynomial in terms of x, the edge of the inner square. Your final answer must be presented as a second degree polynomial in the form $A(x) = ax^2 + bx + c$.
b. Given that the edge of the inner square is 4 inches, use the polynomial in part (a) to determine the area of the shaded region.

55) A rectangular garden is surrounded by a uniform border of lawn measuring x units wide. The entire rectangular plot measures 31 by 29 feet.

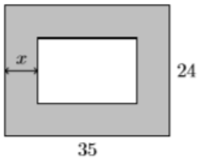


a. Find the area of the interior rectangular garden as a polynomial in terms of x. Your final answer must be presented as a second degree polynomial in the form $A(x) = ax^2 + bx + c$.
b. Given that the width of the border is 9.3 feet, use the polynomial in part (a) to determine the area of the interior rectangular garden.

Answer

$899 - 120x + 4x^2$, 128.96 square feet

56) A rectangular garden is surrounded by a uniform border of lawn measuring x units wide. The entire rectangular plot measures 35 by 24 feet.



a. Find the area of the interior rectangular garden as a polynomial in terms of x. Your final answer must be presented as a second degree polynomial in the form $A(x) = ax^2 + bx + c$.

b. Given that the width of the border is 1.5 feet, use the polynomial in part (a) to determine the area of the interior rectangular garden.

5.7: Special Products

In Exercises 1-12, use the FOIL shortcut as in Example 5.7.3 and Example 5.7.4 to multiply the given binomials.

1) $(5x+2)(3x+4)$

Answer

$15x^2+26x+8$

2) $(5x+2)(4x+3)$

3) $(6x-3)(5x+4)$

Answer

$30x^2+9x-12$

4) $(6x-2)(4x+5)$

5) $(5x-6)(3x-4)$

Answer

$15x^2-38x+24$

6) $(6x-4)(3x-2)$

7) $(6x-2)(3x-5)$

Answer

$18x^2-36x+10$

8) $(2x-3)(6x-4)$

9) $(6x+4)(3x+5)$

Answer

$18x^2+42x+20$

10) $(3x+2)(4x+6)$

11) $(4x-5)(6x+3)$

Answer

$24x^2-18x-15$

12) $(3x-5)(2x+6)$

In Exercises 13-20, use the difference of squares shortcut as in Example 5.7.5 to multiply the given binomials.

13) $(10x-12)(10x+12)$

Answer

$100x^2-144$

14) $(10x-11)(10x+11)$

15) $(6x+9)(6x-9)$

Answer

$36x^2 - 81$

16) $(9x+2)(9x-2)$

17) $(3x+10)(3x-10)$

Answer

$9x^2 - 100$

18) $(12x+12)(12x-12)$

19) $(10x-9)(10x+9)$

Answer

$100x^2 - 81$

20) $(4x-6)(4x+6)$

In Exercises 21-28, use the squaring a binomial shortcut as in Example 5.7.8 to expand the given expression.

21) $(2x+3)^2$

Answer

$4x^2 + 12x + 9$

22) $(8x+9)^2$

23) $(9x-8)^2$

Answer

$81x^2 - 144x + 64$

24) $(4x-5)^2$

25) $(7x+2)^2$

Answer

$49x^2 + 28x + 4$

26) $(4x+2)^2$

27) $(6x-5)^2$

Answer

$36x^2 - 60x + 25$

28) $(4x-3)^2$

In Exercises 29-76, use the appropriate shortcut to multiply the given binomials.

29) $(11x-2)(11x+2)$

Answer

$121x^2 - 4$

30) $(6x-7)(6x+7)$

31) $(7r-5t)^2$

Answer

$49r^2 - 70rt + 25t^2$

32) $(11u - 9w)^2$

33) $(5b + 6c)(3b - 2c)$

Answer

$15b^2 + 8bc - 12c^2$

34) $(3r + 2t)(5r - 3t)$

35) $(3u + 5v)(3v - 5v)$

Answer

$9u^2 - 25v^2$

36) $(11a + 4c)(11a - 4c)$

37) $\left(9b^3 + 10c^5\right)\left(9b^3 - 10c^5\right)$

Answer

$81b^6 - 100c^{10}$

38) $\left(9r^5 + 7t^2\right)\left(9r^5 - 7t^2\right)$

39) $(9s - 4t)(9s + 4t)$

Answer

$81s^2 - 16t^2$

40) $(12x - 7y)(12x + 7y)$

41) $(7x - 9y)(7x + 9y)$

Answer

$49x^2 - 81y^2$

42) $(10r - 11t)(10r + 11t)$

43) $(6a - 6b)(2a + 3b)$

Answer

$12a^2 + 6ab - 18b^2$

44) $(6r - 5t)(2r + 3t)$

45) $(10x - 10)(10x + 10)$

Answer

$100x^2 - 100$

46) $(12x - 8)(12x + 8)$

47) $(4a + 2b)(6a - 3b)$

Answer

$24a^2 - 6b^2$

48) $(3b + 6c)(2b - 4c)$

49) $(5b - 4c)(3b + 2c)$

Answer

$15b^2 - 2bc - 8c^2$

50) $(3b - 2c)(4b + 5c)$

51) $(4b - 6c)(6b - 2c)$

Answer

$24b^2 - 44bc + 12c^2$

52) $(4y - 4z)(5y - 3z)$

53) $\left(11r^5 + 9t^2\right)^2$

Answer

$121r^{10} + 198r^5t^2 + 81t^4$

54) $\left(11x^3 + 10z^5\right)^2$

55) $(4u - 4v)(2u - 6v)$

Answer

$8u^2 - 32uv + 24v^2$

56) $(4u - 5w)(5u - 6w)$

57) $\left(8r^4 + 7t^5\right)^2$

Answer

$64r^8 + 112r^4t^5 + 49t^{10}$

58) $\left(2x^5 + 5y^2\right)^2$

59) $(4r + 3t)(4r - 3t)$

Answer

$16r^2 - 9t^2$

60) $(3r + 4s)(3r - 4s)$

61) $(5r + 6t)^2$

Answer

$25r^2 + 60rt + 36t^2$

62) $(12v + 5w)^2$

63) $(3x - 4)(2x + 5)$

Answer

$6x^2 + 7x - 20$

64) $(5x - 6)(4x + 2)$

65) $(6b + 4c)(2b + 3c)$

Answer

$12b^2+26bc+12c^2$

66) $(3v+6w)(2v+4w)$

67) $\left(11u^2+8w^3\right)\left(11u^2-8w^3\right)$

Answer

$121u^4-64w^6$

68) $\left(3u^3+11w^4\right)\left(3u^3-11w^4\right)$

69) $(4y+3z)^2$

Answer

$16y^2+24yz+9z^2$

70) $(11b+3c)^2$

71) $(7u-2v)^2$

Answer

$49u^2-28uv+4v^2$

72) $(4b-5c)^2$

73) $(3v+2w)(5v+6w)$

Answer

$15v^2+28vw+12w^2$

74) $(5y+3z)(4y+2z)$

75) $(5x-3)(6x+2)$

Answer

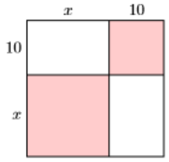
$30x^2-8x-6$

76) $(6x-5)(3x+2)$

For each of the following figure, compute the area of the square using two methods.

a. **Find the area by summing the areas of its parts (see Example 5.5.7).**
b. **Find the area by squaring the side of the square using the squaring a binomial shortcut.**

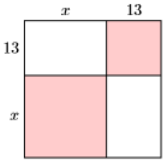
77)



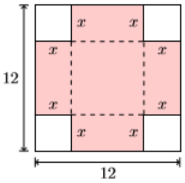
Answer

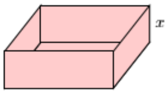
$A=x^2+20x+100$

78)



79) A square piece of cardboard measures 12 inches on each side. Four squares, each having a side of x inches, are cut and removed from each of the four corners of the square piece of cardboard. The sides are then folded up along the dashed lines to form a box with no top.





a. Find the volume of the box as a function of x, the measure of the side of each square cut from the four corner of the original piece of cardboard. Multiply to place your answer in standard polynomial form, simplifying your answer as much as possible.
b. Use the resulting polynomial to determine the volume of the box if squares of length 1.25 inches are cut from each corner of the original piece of cardboard. Round your answer to the nearest cubic inch.

Answer

a. $V(x) = 144x - 48x^2 + 4x^3$
b. $V(1.25) \approx 113$ cubic inches

80) Consider again the box formed in Exercise 79.

a. Find the surface area of the box as a function of x, the measure of the side of each square cut from the four corner of the original piece of cardboard. Multiply to place your answer in standard polynomial form, simplifying your answer as much as possible.
b. Use the resulting polynomial to determine the surface area of the box if squares of length 1.25 inches are cut from each corner of the original piece of cardboard. Round your answer to the nearest square inch.

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CHAPTER OVERVIEW

6: Factoring

6.1: The Greatest Common Factor
6.2: Solving Nonlinear Equations
6.3: Factoring ax² + bx + c when a =1
6.4: Factoring ax² + bx + c when a≠1
6.5: Factoring Special Forms
6.6: Factoring Strategy
6.7: Applications of Factoring
6.E: Factoring (Exercises)

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6.1: The Greatest Common Factor

We begin this section with definitions of factors and divisors. Because $24 = 2 \cdot 12$, both 2 and 12 are factors of 24. However, note that 2 is also a divisor of 24, because when you divide 24 by 2 you get 12, with a remainder of zero. Similarly, 12 is also a divisor of 24, because when you divide 24 by 12 you get 2, with a remainder of zero.

Definitions: Factors and Divisors

Suppose m and n are integers. Then m is a divisor (factor) of n if and only if there exists another integer k so that $n = m \cdot k$.

The words divisor and factor are equivalent. They have the same meaning.

Example $6.1.1$

List the positive divisors (factors) of 24.

Solution

First, list all possible ways that we can express 24 as a product of two positive integers:

$$24 = 1 \cdot 24 \quad \text{or} \quad 24 = 2 \cdot 12 \quad \text{or} \quad 24 = 3 \cdot 8 \quad \text{or} \quad 24 = 4 \cdot 6$$

Therefore, the positive divisors (factors) of 24 are $1, 2, 3, 4, 6, 8,$ and 24.

Exercise $6.1.1$

List the positive divisors of 18.

Answer

$1, 2, 3, 6, 9,$ and 18

Example $6.1.2$

List the positive divisors (factors) that 36 and 48 have in common.

Solution

First, list all positive divisors (factors) of 36 and 48 separately, then box the divisors that are in common.

$$\text{Divisors of } 36 \text{ are: } [1], [2], [3], [4], [6], 9, [12], 18, 36$$

$$\text{Divisors of } 48 \text{ are: } [1], [2], [3], [4], [6], 8, [12], 16, 24, 48$$

Therefore, the common positive divisors (factors) of 36 and 48 are $1, 2, 3, 4, 6,$ and 12.

Exercise $6.1.2$

List the positive divisors that 40 and 60 have in common.

Answer

$1, 2, 4, 5, 10,$ and 20

Definition: Greatest Common Divisor

The **greatest common divisor (factor)** of a and b is the largest positive number that divides evenly (no remainder) both a and b. The **greatest common divisor** of a and b is denoted by the symbolism $\text{GCD}(a, b)$. We will also use the abbreviation $\text{GCF}(a, b)$ to represents the greatest common factor of a and b.

Remember, greatest common divisor and greatest common factor have the same meaning. In Example $6.1.2$, we listed the common positive divisors of 36 and 48. The largest of these common divisors was 12. Hence, the greatest common divisor (factor) of 36 and

48 is 12, written $\text{GCD}(36, 48) = 12$.

With smaller numbers, it is usually easy to identify the greatest common divisor (factor).

Example 6.1.3

State the greatest common divisor (factor) of each of the following pairs of numbers:

a. 18 and 24
b. 30 and 40
c. 16 and 24

Solution

In each case, we must find the largest possible positive integer that divides evenly into both the given numbers.

a. The largest positive integer that divides evenly into both 18 and 24 is 6. Thus, $\text{GCD}(18, 24) = 6$.
b. The largest positive integer that divides evenly into both 30 and 40 is 10. Thus, $\text{GCD}(30, 40) = 10$.
c. The largest positive integer that divides evenly into both 16 and 24 is 8. Thus, $\text{GCD}(16, 24) = 8$.

Exercise 6.1.3

State the greatest common divisor of 36 and 60.

Answer

12

With larger numbers, it is harder to identify the greatest common divisor (factor). However, prime factorization will save the day!

Example 6.1.4

Find the greatest common divisor (factor) of 360 and 756.

Solution

Prime factor 360 and 756, writing your answer in exponential form.

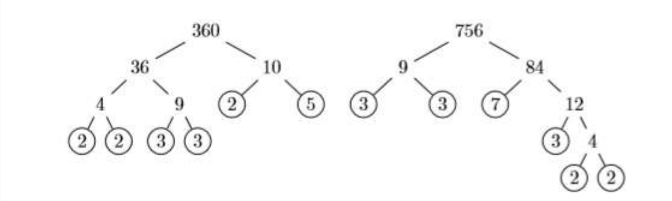


Figure $6.1.1$

Thus:

$$\begin{aligned} 360 &= 2^3 \cdot 3^2 \cdot 5 \\ 756 &= 2^2 \cdot 3^3 \cdot 7 \end{aligned}$$

Note

To find the greatest common divisor (factor), list each factor that appears in common to the highest power that appears in common.

In this case, the factors 2 and 3 appear in common, with 22 being the highest power of 2 and 32 being the highest power of 3 that appear in common. Therefore, the greatest common divisor of 360 and 756 is:

$$\begin{aligned} \text{GCD}(360, 756) &= 2^2 \cdot 3^2 \\ &= 4 \cdot 9 \\ &= 36 \end{aligned}$$

Therefore, the greatest common divisor (factor) is $\text{GCD}(360, 756) = 36$. Note what happens when we write each of the given numbers as a product of the greatest common factor and a second factor:

$$\begin{aligned} 360 &= 36 \cdot 10 \\ 756 &= 36 \cdot 21 \end{aligned}$$

In each case, note how the second second factors (10 and 21) contain no additional common factors.

Exercise 6.1.4

Find the greatest common divisor of 120 and 450.

Answer

30

Finding the Greatest Common Factor of Monomials

Example $6.1.4$ reveals the technique used to find the greatest common factor of two or more monomials.

Finding the GCF of two or more monomials

To find the greatest common factor of two or more monomials, proceed as follows:

1. Find the greatest common factor (divisor) of the coefficients of the given monomials. Use prime factorization if necessary.
2. List each variable that appears in common in the given monomials.
3. Raise each variable that appears in common to the highest power that appears in common among the given monomials.

Example 6.1.5

Find the greatest common factor of $6x^3y^3$ and $9x^2y^5$.

Solution

To find the GCF of $6x^3y^3$ and $9x^2y^5$, we note that:

1. The greatest common factor (divisor) of 6 and 9 is 3.
2. The monomials $6x^3y^3$ and $9x^2y^5$ have the variables x and y in common.
3. The highest power of x in common is x^2. The highest power of y in common is y^3.

Thus, the greatest common factor is $\text{GCF}\left(6x^3y^3, 9x^2y^5\right) = 3x^2y^3$. Note what happens when we write each of the given monomials as a product of the greatest common factor and a second monomial:

$$\begin{aligned} 6x^3y^3 &= 3x^2y^3 \cdot 2x \\ 9x^2y^5 &= 3x^2y^3 \cdot 3y \end{aligned}$$

Observe that the set of second monomial factors ($2x$ and $3y$) contain no additional common factors.

Exercise 6.1.5

Find the greatest common factor of $16xy^3$ and $12x^4y^2$.

Answer

$4xy^2$

Example 6.1.6

Find the greatest common factor of $12x^4$, $18x^3$, and $30x^2$.

Solution

To find the GCF of $12x^4$, $18x^3$, and $30x^2$, we note that:

1. The greatest common factor (divisor) of 12, 18, and 30 is 6.
2. The monomials $12x^4$, $18x^3$, and $30x^2$ have the variable x in common.
3. The highest power of x in common is x^2.

Thus, the greatest common factor is $\mathrm{GCF}\left(12x^4, 18x^3, 30x^2\right) = 6x^2$. Note what happens when we write each of the given monomials as a product of the greatest common factor and a second monomial:

$$\begin{aligned} 12x^4 &= \color{red}{6x^2} \cdot 2x^2 \\ 18x^3 &= \color{red}{6x^2} \cdot 3x \\ 30x^2 &= \color{red}{6x^2} \cdot 5 \end{aligned}$$

Observe that the set of second monomial factors ($2x^2$, $3x$, and 5) contain no additional common factors.

Exercise 6.1.6

Find the greatest common factor of $6y^3$, $15y^2$, and $9y^5$.

Answer

$3y^2$

Factor Out the GCF

In Chapter 5, we multiplied a monomial and polynomial by distributing the monomial times each term in the polynomial.

$$\begin{aligned} \color{red}{2x(}3x^2 + 4x - 7\color{red}{)} &= \color{red}{2x} \cdot 3x^2 + \color{red}{2x} \cdot 4x - \color{red}{2x} \cdot 7 \\ &= 6x^3 + 8x^2 - 14x \end{aligned}$$

In this section we reverse that multiplication process. We present you with the final product and ask you to bring back the original multiplication problem. In the case $6x^3 + 8x^2 - 14x$, the greatest common factor of $6x^3$, $8x^2$, and $14x$ is $2x$. We then use the distributive property to factor out $2x$ from each term of the polynomial.

$$\begin{aligned} 6x^3 + 8x^2 - 14x &= \color{red}{2x} \cdot 3x^2 + \color{red}{2x} \cdot 4x - \color{red}{2x} \cdot 7 \\ &= \color{red}{2x(}3x^2 + 4x - 7\color{red}{)} \end{aligned}$$

Factoring

Factoring is “unmultiplying.” You are given the product, then asked to find the original multiplication problem.

First rule of factoring

If the terms of the given polynomial have a greatest common factor (GCF), then factor out the GCF.

Let’s look at a few examples that factor out the GCF.

Example 6.1.7

Factor: $6x^2 + 10x + 14$

Solution

The greatest common factor (GCF) of $6x^2$, $10x$ and 14 is 2. Factor out the GCF.

$$\begin{aligned} 6x^2 + 10x + 14x &= \color{red}{2} \cdot 3x^2 + \color{red}{2} \cdot 5x + \color{red}{2} \cdot 7 \\ &= \color{red}{2(}3x^2 + 5x + 7\color{red}{)} \end{aligned}$$

Checking your work

Every time you factor a polynomial, remultiply to check your work.

Check: Multiply. Distribute the 2.

$$\color{red}{2(}3x^2 + 5x + 7\color{red}{)} = \color{red}{2} \cdot 3x^2 + \color{red}{2} \cdot 5x + \color{red}{2} \cdot 7$$

$$= 6x^2 + 10x + 14$$

That’s the original polynomial, so we factored correctly.

Exercise 6.1.7

Factor: $9y^2 - 15y + 12$

Answer

$3\left(3y^2 - 5y + 4\right)$

Example 6.1.8

Factor: $12y^5 - 32y^4 + 8y^2$

Solution

The greatest common factor (GCF) of $12y^5$, $32y^4$ and $8y^2$ is $4y^2$. Factor out the GCF.

$$\begin{aligned} 12y^5 - 32y^4 + 8y^2 &= 4y^2 \cdot 3y^3 - 4y^2 \cdot 8y^2 + 4y^2 \cdot 2 \\ &= 4y^2(3y^3 - 8y^2 + 2) \end{aligned}$$

Check: Multiply. Distribute the monomial $4y^2$.

$$\begin{aligned} 4y^2(3y^3 - 8y^2 + 2) &= 4y^2 \cdot 3y^3 - 4y^2 \cdot 8y^2 + 4y^2 \cdot 2 \\ &= 12y^5 - 32y^4 + 8y^2 \end{aligned}$$

That’s the original polynomial. We have factored correctly.

Exercise 6.1.8

Factor: $8x^6 + 20x^4 - 24x^3$

Answer

$4x^3\left(2x^3 + 5x - 6\right)$

Example 6.1.9

Factor: $12a^3b + 24a^2b^2 + 12ab^3$

Solution

The greatest common factor (GCF) of $12a^3b$, $24a^2b^2$ and $12ab^3$ is $12ab$. Factor out the GCF.

$$\begin{aligned} 12a^3b + 24a^2b^2 + 12ab^3 &= 12ab \cdot a^2 - 12ab \cdot 2ab + 12ab \cdot b^2 \\ &= 12ab(a^2 + 2ab + b^2) \end{aligned}$$

Check: Multiply. Distribute the monomial $12ab$.

$$\begin{aligned} 12ab(a^2 + 2ab + b^2) &= 12ab \cdot a^2 - 12ab \cdot 2ab + 12ab \cdot b^2 \\ &= 12a^3b + 24a^2b^2 + 12ab^3 \end{aligned}$$

That’s the original polynomial. We have factored correctly.

Exercise 6.1.9

Factor: $15s^2t^4 + 6s^3t^2 + 9s^2t^2$

Answer

$3s^2t^2\left(5t^2 + 2s + 3\right)$

Speeding Things Up a Bit

Eventually, after showing your work on a number of examples such as those in Examples $6.1.7$, $6.1.8$, and $6.1.9$, you'll need to learn how to perform the process mentally.

Example $6.1.10$

Factor each of the following polynomials:

a. $24x+32$
b. $5x^3-10x^2-10x$
c. $2x^4y+2x^3y^2-6x^2y^3$

Solution

In each case, factor out the greatest common factor (GCF):

a. The GCF of $24x$ and 32 is 8. Thus,

$$24x+32=8(3x+4)$$

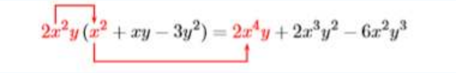
b. The GCF of $5x^3$, $10x^2$, and $10x$ is $5x$. Thus:

$$5x^3-10x^2-10x=5x\left(x^2-2x-2\right)$$

c. The GCF of $2x^4y$, $2x^3y^2$, and $6x^2y^3$ is $2x^2y$. Thus:

$$2x^4y+2x^3y^2-6x^2y^3=2x^2y\left(x^2+xy-3y^2\right)$$

As you speed things up by mentally factoring out the \(\mathrm{GCF}\), it is even more important that you check your results. The check can also be done mentally. For example, in checking the third result, mentally distribute $2x^2y$ times each term of $x^2+xy-3y^2$. Multiplying $2x^2y$ times the first term x^2 produces $2x^4y$, the first term in the original polynomial.



Continue in this manner, mentally checking the product of $2x^2y$ with each term of $x^2+xy-3y^2$, making sure that each result agrees with the corresponding term of the original polynomial.

Exercise $6.1.10$

Factor: $18p^5q^4-30p^4q^5+42p^3q^6$

Answer

$6p^3q^4\left(3p^2-5pq+7q^2\right)$

Remember that the distributive property allows us to pull the GCF out in front of the expression or to pull it out in back. In symbols:

$$ab+ac=a(b+c)\quad\text{or}\quad ba+ca=(b+c)a$$

Example $6.1.11$

Factor: $2x(3x+2)+5(3x+2)$

Solution

In this case, the greatest common factor (GCF) is $3x+2$.

$$\begin{aligned}2x(3x+2)+5(3x+2)&=2x\cdot(3x+2)+5\cdot(3x+2)\\&=(2x+5)(3x+2)\end{aligned}$$

Because of the commutative property of multiplication, it is equally valid to pull the GCF out in front.

$$2x(3x+2)+5(3x+2)=2x\cdot(3x+2)+5\cdot(3x+2)$$

$$= (3x+2)(2x+5) \tag{6.1.1}$$

Note that the order of factors differs from the first solution, but because of the commutative property of multiplication, the order does not matter. The answers are the same.

Exercise 6.1.11

Factor: $3x^2(4x-7)+8(4x-7)$

Answer

$\left(3x^2+8\right)(4x-7)$

Example 6.1.12

Factor: $15a(a+b)-12(a+b)$

Solution

In this case, the greatest common factor (GCF) is $3(a+b)$.

$$\begin{aligned} 15a(a+b)-12(a+b) &= 3(a+b)\cdot 5a-3(a+b)\cdot 4 \\ &= 3(a+b)(5a-4) \end{aligned}$$

Alternate solution:

It is possible that you might fail to notice that 15 and 12 are divisible by 3, factoring out only the common factor $a+b$.

$$\begin{aligned} 15a(a+b)-12(a+b) &= 15a\cdot(a+b)-12\cdot(a+b) \\ &= (15a-12)(a+b) \end{aligned}$$

However, you now need to notice that you can continue, factoring out 3 from both $15a$ and 12.

$$=3(5a-4)(a+b)$$

Note that the order of factors differs from the first solution, but because of the commutative property of multiplication, the order does not matter. The answers are the same.

Exercise 6.1.12

Factor: $24m(m-2n)+20(m-2n)$

Answer

$4(6m+5)(m-2n)$

Factoring by Grouping

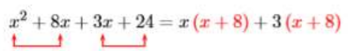
The final factoring skill in this section involves four-term expressions. The technique for factoring a four-term expression is called *factoring by grouping*.

Example 6.1.13

Factor by grouping: $x^2+8x+3x+24$

Solution

We "group" the first and second terms, noting that we can factor an x out of both of these terms. Then we "group" the third and fourth terms, noting that we can factor 3 out of both of these terms.



$$x^2+8x+3x+24=x(x+8)+3(x+8)$$

Now we can factor $x+8$ out of both of these terms.

$$(x+3)\color{red}{(x+8)}$$

Exercise 6.1.13

Factor by grouping: $x^2-6x+2x-12$

Answer

$(x+2)(x-6)$

Let’s try a grouping that contains some negative signs.

Example 6.1.14

Factor by grouping: $x^2+4x-7x-28$

Solution

We “group” the first and second terms, noting that we can factor x out of both of these terms. Then we “group” the third and fourth terms, then try to factor a 7 out of both these terms.

$$\underbrace{x^2+4x}\underbrace{-7x-28}=x\color{red}{(x+4)}\color{black}{+7}\color{red}{(-x-4)}$$

This does not lead to a common factor. Let’s try again, this time factoring a -7 out of the third and fourth terms.

$$\underbrace{x^2+4x}\underbrace{-7x-28}=x\color{red}{(x+4)}\color{black}{-7}\color{red}{(x+4)}$$

That worked! We now factor out a common factor $x+4$.

$$(x-7)\color{red}{(x+4)}$$

Exercise 6.1.14

Factor by grouping: $x^2-5x-4x+20$

Answer

$(x-4)(x-5)$

Let’s increase the size of the numbers a bit.

Example 6.1.15

Factor by grouping: $6x^2-8x+9x-12$

Solution

Note that we can factor $2x$ out of the first two terms and 3 out of the second two terms.

$$\underbrace{6x^2-8x}\underbrace{+9x-12}=2x\color{red}{(3x-4)}\color{black}{+3}\color{red}{(3x-4)}$$

Now we have a common factor $3x-4$ which we can factor out.

$$(2x+3)\color{red}{(3x-4)}$$

Exercise 6.1.15

Factor by grouping: $15x^2+9x+10x+6$

Answer

$(3x+2)(5x+3)$

As the numbers get larger and larger, you need to factor out the (GCF) from each grouping. If not, you won't get a common factor to finish the factoring.

Example $6.1.16$

Factor by grouping: $24x^2 - 32x - 45x + 60$

Solution

Suppose that we factor $8x$ out of the first two terms and -5 out of the second two terms.

$$\underbrace{24x^2 - 32x}_{} \underbrace{- 45x + 60}_{} = 8x(3x-4) - 5(9x-12)$$

That did not work, as we don't have a common factor to complete the factoring process. However, note that we can still factor out a 3 from $9x - 12$. As we've already factored out a 5, and now we see can factor out an additional 3, this means that we should have factored out 3 times 5, or 15, to begin with. Let's start again, only this time we'll factor 15 out of the second two terms.

$$\underbrace{24x^2 - 32x}_{} \underbrace{- 45x + 60}_{} = 8x(3x-4) - 15(3x-4)$$

Beautiful! We can now factor out $3x - 4$.

$$(8x - 15)(3x - 4)$$

Exercise $6.1.16$

Factor by grouping: $36x^2 - 84x + 15x - 35$

Answer

$(12x + 5)(3x - 7)$

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6.2: Solving Nonlinear Equations

We begin by introducing a property that will be used extensively in this and future sections.

The Zero Product Property

If the product of two or more numbers equals zero, then at least one of the numbers must equal zero. That is, if

$$ab=0$$

then

$$a=0 \text{ or } b=0$$

Let's use the *zero product property* to solve a few equations.

Example $6.2.1$

Solve for x: $(x+3)(x-5)=0$

Solution

The product of two factors equals zero.

$$(x+3)(x-5)=0$$

Hence, at least one of the factors must equal zero. Using the zero product property, set each factor equal to zero, then solve the resulting equations for x.

$$\begin{aligned} x+3&=0 \\ x&=-3 \end{aligned}$$

or

$$\begin{aligned} x-5&=0 \\ x&=5 \end{aligned}$$

Hence, the solutions are $x=-3$ and $x=5$

Check:

Check that each solution satisfies the original equation.

Substitute -3 for x:

$$\begin{aligned} (x+3)(x-5)&=0 \\ (-3+3)(-3-5)&=0 \\ (0)(-8)&=0 \\ 0&=0 \end{aligned}$$

Substitute 5 for x:

$$\begin{aligned} (x+3)(x-5)&=0 \\ (5+3)(5-5)&=0 \\ (8)(0)&=0 \\ 0&=0 \end{aligned}$$

Because each check produces a true statement, both $x=-3$ and $x=5$ are solutions of $(x+3)(x-5)=0$.

Exercise $6.2.1$

Solve for x: $(x-7)(x-2)=0$

Answer

$7, 2$

The zero product property also works equally well if more than two factors are present. For example, if $abc = 0$, then either $a = 0$ or $b = 0$ or $c = 0$. Let's use this idea in the next example.

Example $6.2.2$

Solve for x: $x(2x+9)(3x-5) = 0$

Solution

The product of three factors equals zero.

$$x(2x+9)(3x-5) = 0$$

Using the zero product property, set each factor equal to zero, then solve the resulting equations for x.

$$x = 0$$

or

$$\begin{aligned} 2x+9 &= 0 \\ 2x &= -9 \\ x &= -\frac{9}{2} \end{aligned}$$

or

$$\begin{aligned} 3x-5 &= 0 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

Hence, the solutions are $x = 0$, $x = -9/2$, and $x = 5/3$. We encourage the reader to check the solution.

Exercise $6.2.2$

Solve for x: $6x(x+4)(5x+1) = 0$

Answer

$0, -4, -1/5$

Linear versus Nonlinear

All of the equations solved in previous chapters were examples of what are called **linear equations**. If the highest power of the variable we are solving for is one, then the graphs involved are lines. Hence the term, linear equation. However, if the power on the variable we are solving for exceeds one, then the graphs involved are curves. Hence the term, nonlinear equation. In this chapter we will learn how to solve nonlinear equations involving polynomials. However, let's first make sure we can recognize the difference between a linear and a nonlinear equation.

Definition: Linear versus Nonlinear Equations

Use the following conditions to determine if an equation is linear or nonlinear.

1. If the highest power of the variable we are solving for is one, then the equation is *linear*.
2. If the highest power of the variable we are solving for is larger than one, then the equation is *nonlinear*.

Example $6.2.3$

If the instruction is "solve for x," classify each of the following equations as linear or nonlinear.

a. $3x-5 = 4-7x$
b. $x^2 = 8x$

Solution

Because the instruction is "solve for x," to determine whether the equation is linear or nonlinear, we identify the largest power of x present in the equation.

a. The highest power of x present in the equation $3x-5=4-7x$ is one. Hence, this equation is linear.
b. The equation $x^2=8x$ contains a power of x higher than one (it contains an x^2). Hence, this equation is nonlinear.

Exercise 6.2.3

Classify the following equation as linear or nonlinear: $2x=x^3-4$

Answer

nonlinear

Now that we can classify equations as either *linear* or *nonlinear*, let's introduce strategies for solving each type, the first of which should already be familiar.

Strategy for solving a linear equation

If an equation is *linear*, start the solution process by moving all terms containing the variable you are solving for to one side of the equation, then move all terms that do not contain the variable you are solving for to the other side of the equation.

Example 6.2.4

Solve for x: $3x-5=4-7x$

Solution

Because the instruction is "solve for x" and we note that the largest power of x present is one, the equation $3x-5=4-7x$ is linear. Hence, the strategy is to move all terms containing x to one side of the equation, then move all the remaining terms to the other side of the equation.

$$\begin{aligned} 3x-5 &= 4-7x && \color{red}{\text{Original equation.}} \\ 3x-5+7x &= 4 && \color{red}{\text{Add } 7x \text{ to both sides.}} \\ 3x+7x &= 4+5 && \color{red}{\text{Add } 5 \text{ to both sides.}} \end{aligned}$$

Note how we have succeeded in moving all terms containing x to one side of the equation and all terms that do not contain x to the other side of the equation.

$$\begin{aligned} 10x &= 9 && \color{red}{\text{Simplify both sides.}} \\ x &= \frac{9}{10} && \color{red}{\text{Divide both sides by 10.}} \end{aligned}$$

Hence, the solution of $3x-5=4-7x$ is $x=9/10$. Readers are encouraged to check this solution.

Exercise 6.2.4

Add exercises text here.

Answer

$1/4$

The situation is much different when the equation is nonlinear.

Strategy for solving a nonlinear equation

If an equation is *nonlinear*, first move everything to one side of the equation, making one side of the equation equal to zero. Continue the solution process by factoring and applying the zero *product property*.

Example 6.2.5

Solve for x: $x^2 = 8x$

Solution

Because the instruction is “solve for x,” and the highest power of x is larger than one, the equation $x^2 = 8x$ is nonlinear. Hence, the strategy requires that we move all terms to one side of the equation, making one side zero.

$$\begin{aligned} x^2 &= 8x && \text{Original equation.} \\ x^2 - 8x &= 0 && \text{Subtract } 8x \text{ from both sides.} \end{aligned}$$

Note how we have succeeded in moving all terms to one side of the equation, making one side equal to zero. To finish the solution, we factor out the GCF on the left-hand side.

$$x(x-8) = 0 \quad \text{Factor out the GCF.}$$

Note that we now have a product of two factors that equals zero. By the zero product property, either the first factor is zero or the second factor is zero.

$$\begin{aligned} x = 0 \quad \text{or} \quad x - 8 &= 0 \\ x &= 8 \end{aligned}$$

Hence, the solutions are $x = 0$ and $x = 8$.

Check:

Check that each solution satisfies the original equation.

$$\begin{gathered} \text{Substitute } 0 \text{ for } x: \\ x^2 = 8x \\ (0)^2 = 8(0) \\ 0 = 0 \\ \\ \text{Subtitute } 8 \text{ for } x: \\ x^2 = 8x \\ (8)^2 = 8(8) \\ 64 = 64 \end{gathered}$$

Note that both results are true statements, guaranteeing that both $x = 0$ and $x = 8$ are solutions of $x^2 = 8x$

Exercise 6.2.5

Solve for x: $x^2 = -5x$

Answer

$0, -5$

Warning!

The following is incorrect!

Consider what would happen if we divided both sides of the equation $x^2 = 8x$ in Example 6.2.5 by x:

$$\begin{gathered} x^2 = 8x \\ \frac{x^2}{x} = \frac{8x}{x} \\ x = 8 \end{gathered}$$

Note that we have lost the second answer found in Example 6.2.5, $x = 0$. This example demonstrates that you should never divide by the variable you are solving for! If you do, and cancellation occurs, you will lose answers.

Let's try solving a nonlinear equation that requires factoring by grouping.

Example $6.2.6$

Solve for x: $6x^2+9x-8x-12=0$

Solution

Because we are solving for x and there is a power of x larger than one, this equation is nonlinear. Hence, the first step is to move everything to one side of the equation, making one side equal to zero. Well that's already done, so let's factor the left-hand side by grouping. Note that we can factor $3x$ out of the first two terms and -4 out of the second two terms.

$$\begin{aligned} 6x^2+9x-8x-12&=0 \\ 3x\,(2x+3)-4\,(2x+3)&=0 \end{aligned}$$

Factor out the common factor $2x+3$.

$$(3x-4)(2x+3)=0$$

We now have a product of two factors that equals zero. Use the zero product property to write:

$$\begin{aligned} 3x-4&=0 \\ 3x&=4 \\ x&=\frac{4}{3} \end{aligned}$$

or

$$\begin{aligned} 2x+3&=0 \\ 2x&=-3 \\ x&=-\frac{3}{2} \end{aligned} \tag{6.2.1}$$

Hence, the solutions are $x=4/3$ and $x=-3/2$.

Check:

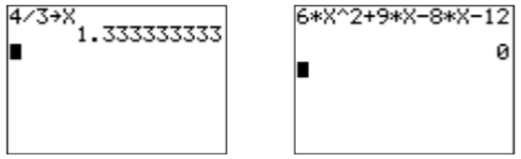
Let's use the graphing calculator to check the solution $x=4/3$. First, store the solution $4/3$ in the variable $\mathbb{X}$ using the following keystrokes (see the first image in Figure $6.2.1$.

4 ÷ 3 STO▷ X,T,θ,n ENTER

Figure $6.2.1$ indicates that the expression $6x^2+9x-8x-12$ equals zero when $x=4/3$.

6 × X,T,θ,n ∧ 2 + 9 × X,T,θ,n −

8 × X,T,θ,n − 1 2 ENTER



```
4/3→X
      1.333333333
```

```
6*X^2+9*X-8*X-12
                0
```

Figure $6.2.1$: Checking the solution $x=4/3$.

Therefore, the solution $x=4/3$ checks. Readers are encouraged to use their graphing calculators to check the second solution, $x=-3/2$.

Exercise $6.2.6$

Solve for x: $5x^2-20x-4x+16=0$

Answer

$4/5, 4$

Using the Graphing Calculator

In this section we will employ two different calculator routines to find the solution of a nonlinear equation. Before picking up the calculator, let’s first use an algebraic method to solve the equation $x^2 = -5x$. The equation is nonlinear, so the first step is to move everything to one side of the equation, making one side equal to zero.

$$\begin{aligned} x^2 &= -5x && \text{Nonlinear. Make one side zero.} \\ x^2 + 5x &= 0 && \text{Add } 5x \text{ to both sides.} \\ x(x+5) &= 0 && \text{Factor out the GCF.} \end{aligned}$$

Use the zero product property, setting each factor equal to zero, then solving the resulting equations for x.

$$x = 0$$

or

$$\begin{aligned} x + 5 &= 0 \\ x &= -5 \end{aligned}$$

Hence, the solutions are $x = 0$ and $x = -5$.

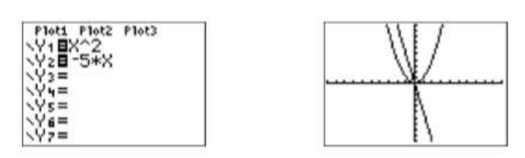
We’ll now use the calculator to find the solutions of $x^2 = -5x$. The first technique employs the **5:intersect** routine on the calculator’s **CALC** menu.

Example 6.2.7

Use the **5:intersect** utility on the graphing calculator to solve the equation $x^2 = -5x$ for x.

Solution

Load the left-hand side of $x^2 = -5x$ in $\mathbb{Y}1$ and the right-hand side in $\mathbb{Y}2$ (see Figure 6.2.2). Selecting **6:ZStandard** from the **ZOOM** menu produces the graphs shown in the image on the right in Figure 6.2.2.

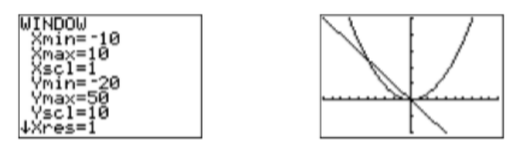


```
Plot1 Plot2 Plot3
\Y1■X^2
\Y2■-5*X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 6.2.2: Sketch the graphs of each side of the equation $x^2 = -5x$.

Note that the graph of $y = x^2$ is a parabola that opens upward, with vertex (turning point) at the origin. This graph reveals why the equation $x^2 = -5x$ is called a nonlinear equation (not all the graphs involved are lines). Next, the graph of $y = -5x$ is a line with slope -5 and y-intercept at the origin.

The two graphs obviously intersect at the origin, but it also appears that there may be another point of intersection that is off the screen. Let’s increase $\mathbb{Y}\mathrm{max}$ in an attempt to reveal the second point of intersection. After some experimentation, the settings shown in the first image in Figure 6.2.3 reveal both points of intersection. Pushing the **GRAPH** button produces the image on the right in Figure 6.2.3.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-20
 Ymax=50
 Yscl=10
↓Xres=1
```

Figure 6.2.3: Adjust the **WINDOW** parameters to reveal both points of intersection.

To find the solutions of the equation $x^2 = -5x$, we must find the coordinates of the points where the graphs of $y = x^2$ and $y = -5x$ intersect. The x-coordinate of each point of intersection will be a solution of the equation $x^2 = -5x$.

- Start by selecting **5:intersect** from the **CALC** menu. When prompted for the “First curve?”, press **ENTER**. When prompted for the “Second curve?”, press **ENTER**. When prompted for a “Guess,” press **ENTER**. The result is the point $(0, 0)$ shown in the image on the left in Figure $6.2.4$.
- Repeat the process a second time. Select **5:intersect** from the **CALC** menu. When prompted for the “First curve?”, press **ENTER**. When prompted for the “Second curve?”, press **ENTER**. When prompted for a “Guess,” use the left-arrow key to move the cursor closer to the leftmost point of intersection, then press **ENTER**. The result is the point $(-5, 25)$ shown in the image on the right in Figure $6.2.4$.

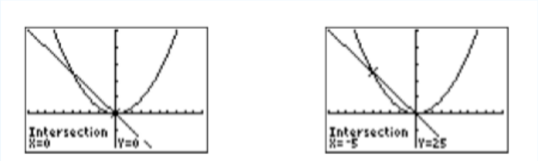


Figure $6.2.4$: Use the 5:intersect utility to find the points of intersection.

Reporting the solution on your homework:

Duplicate the image in your calculator’s viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

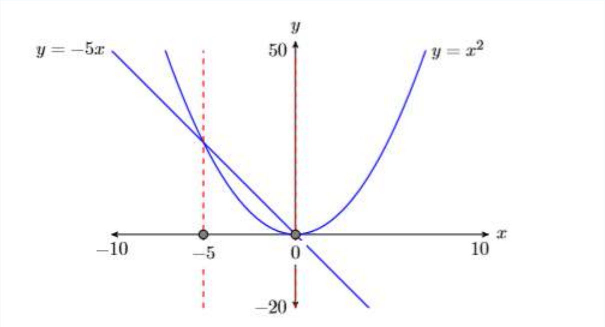


Figure $6.2.5$: Reporting your graphical solution on your homework.

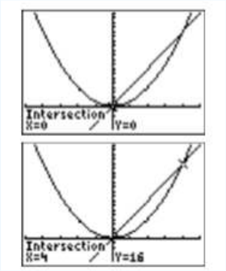
- Label the horizontal and vertical axes with x and y, respectively (see Figure $6.2.5$).
- Place your **WINDOW** parameters at the end of each axis (see Figure $6.2.5$).
- Label each graph with its equation (see Figure $6.2.5$).
- Drop dashed vertical lines through each point of intersection. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 = -5x$ (see Figure $6.2.5$).

Hence, the solutions of $x^2 = -5x$ are $x = -5$ and $x = 0$. Note now these match the solutions found using the algebraic technique.

Exercise $6.2.7$

Use the **5:intersect** utility on the graphing calculator to solve the equation $x^2 = 4x$ for x.

Answer



Before demonstrating a second graphing calculator technique for solving nonlinear equations, let's take a moment to recall the definition of a zero of a function, which was first presented in Chapter 5, Section 3.

Zeros and x-intercepts

The points where the graph of f crosses the x-axis are called the x-intercepts of the graph of f. The x-value of each x-intercept is called a zero of the function f.

We'll now employ the **2:zero** utility from the **CALC** menu to find the solutions of the equation $x^2 = -5x$.

Example $6.2.8$

Use the **2:zero** utility on the graphing calculator to solve the equation $x^2 = -5x$ for x.

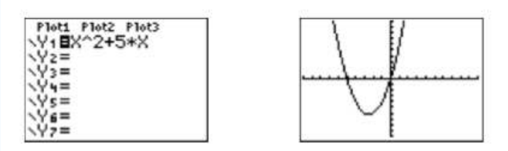
Solution

First, make one side of the equation equal to zero.

$$\begin{aligned} x^2 &= -5x && \text{Make one side zero.} \\ x^2 + 5x &= 0 && \text{Add } 5x \text{ to both sides.} \end{aligned}$$

To determine the values of x that make $x^2 + 5x = 0$, we must locate the points where the graph of $f(x) = x^2 + 5x$ crosses the x-axis. These points are the x-intercepts of the graph of f and the x-values of these points are the zeros of the function f.

Load the function $f(x) = x^2 + 5x$ in $\mathtt{Y1}$, then select **6:ZStandard** to produce the image in Figure $6.2.6$. Note that the graph of f has two x-intercepts, and the x-values of each of these points are the zeros of the function f.



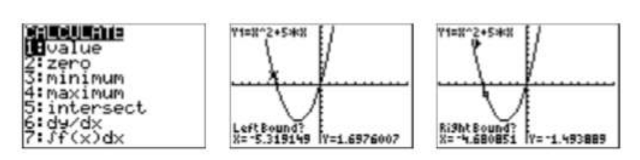
```
Plot1  Plot2  Plot3
\Y1■X^2+5*X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $6.2.6$: Sketch the graph of $p(x) = x^2 + 5x$.

Note

It's often easier to find the solutions of a nonlinear equation by making one side zero and identifying where the graph of the resulting function crosses the x-axis.

Select **2:zero** from the **CALC** menu (see Figure \(\PageIndex{7}\)).



```
Y1=X^2+5*X
Left Bound?
X=⁻5.319149  Y=1.6976007
```

```
Y1=X^2+5*X
Right Bound?
X=⁻4.680851  Y=⁻1.493889
```

Figure $6.2.7$: Setting the left and right bounds when using the **2:zero** utility to find the x-intercepts of the graph of $f(x) = x^2 + 5x$.

- The calculator responds by asking for a "Left Bound?" Use the left-arrow key to move the cursor so that it lies to the left of the x-intercept near $(-5, 0)$ (see the second image in Figure $6.2.7$), then press the **ENTER** key.
- The calculator responds by asking for a "Right Bound?" Move the cursor so that is slightly to the right of the x-intercept near $(-5, 0)$ (see the third image Figure $6.2.7$), then press the **ENTER** key.
- The calculator responds by asking for a "Guess?" Note the two triangular marks near the top of the viewing window in the first image in Figure $6.2.8$ that mark the left- and right-bounds. As long as you place the cursor so that the x-value of the cursor location lies between these two marks, you've made a valid guess. Because the cursor already lies between these two marks, we usually leave it where it is and press the **ENTER** key.

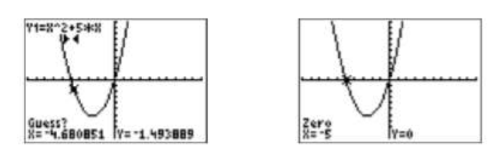


Figure 6.2.8: *Setting the right bound and making a guess.*

After making your guess and pressing the **ENTER** key, the calculator proceeds to find an approximation of the x-intercept that lies between the left- and right-bounds previously marked (see the second image in Figure 6.2.8. Hence, this x-intercept is $(-5, 0)$, making -5 a zero of $f(x) = x^2 + 5x$ and a solution of the equation $x^2 + 5x = 0$.

We'll leave it to our readers to repeat the **2:zero** process to find the second zero at the origin.

Reporting the solution on your homework:

Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 6.2.9).
- Place your WINDOW parameters at the end of each axis (see Figure 6.2.9).
- Label each graph with its equation (see Figure 6.2.9).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of each x-intercept. These are the solutions of the equation $x^2 = -5x$ (see Figure 6.2.9).

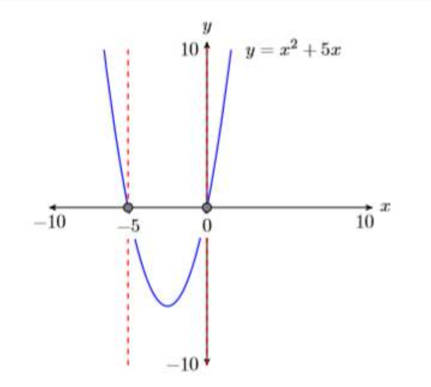


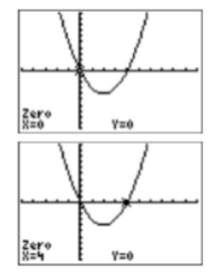
Figure 6.2.9: Reporting your graphical solution on your homework.

Hence, the solutions of $x^2 = -5x$ are $x = -5$ and $x = 0$. Note how nicely this agrees with the solutions found using the algebraic technique and the solutions found using the **5:intersect** utility in Example 6.2.7.

Exercise 6.2.8

Use the **2:zero** utility on the graphing calculator to solve the equation $x^2 = 4x$ for x.

Answer



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6.3: Factoring ax² + bx + c when a =1

In this section we concentrate on learning how to factor trinomials having the form $ax^2 + bx + c$ when $a = 1$. The first task is to make sure that everyone can properly identify the coefficients a, b, and c.

Example 6.3.1

Compare $x^2 - 8x - 9$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Solution

Align the trinomial $x^2 \; - 8x \; - 9$ with the standard form $ax^2 \; + bx \; + c$, then compare coefficients. Note that the understood coefficient of x^2 is 1.

$$\begin{array}{c} ax^2 + bx + c \\ {\color{red}1}x^2 - 8x - 9 \end{array}$$

We see that $a = 1$, $b = -8$, and $c = -9$. Because the leading coefficient is 1, this is the type of trinomial that we will learn how to factor in this section.

Exercise 6.3.1

Compare $2x^2 + 5x - 3$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Answer

$a = 2$, $b = 5$, $c = -3$

Example 6.3.2

Compare $-40 + 6x^2 - x$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Solution

First, arrange $-40 + 6x^2 - x$ in descending powers of x, then align it with the standard form $ax^2 \; + bx \; + c$ and compare coefficients. Note that the understood coefficient of x is -1.

$$\begin{array}{c} ax^2 + bx + c \\ 6x^2 - {\color{red}1}x - 40 \end{array}$$

We see that $a = 6$, $b = -1$, and $c = -40$. Because the leading coefficient is 6, we will have to wait until we learn about factoring $ax^2 + bx + c$ when $a \neq 1$ in section 6.4 before learning how to factor this trinomial.

Exercise 6.3.2

Compare $3x + 9 - 7x^2$ with the form $ax^2 + bx + c$ and identify the coefficients a, b, and c.

Answer

$a = -7$, $b = 3$, $c = 9$

In this section, the leading coefficient must equal 1. Our work in this section will focus only on trinomials of the form $x^2 + bx + c$, that is, the form $ax^2 + bx + c$ where $a = 1$.

The ac-Method

We are now going to introduce a technique called the ac-method (or ac-test) for factoring trinomials of the form $ax^2 + bx + c$ when $a = 1$. In the upcoming section 6.4 about factoring $ax^2 + bx + c$ when $a \neq 1$, we will see that this method can also be employed when $a \neq 1$, with one minor exception. But for the remainder of this section, we focus strictly on trinomials whose leading coefficient is 1.

Let’s begin by finding the following product:

$$\begin{aligned}(x+12)(x-4) &= x(x-4)+12(x-4) && \text{Apply the distributive property.}\\ &= x^2-4x+12x-48 && \text{Distribute again.}\\ &= x^2+8x-48 && \text{Simplify.}\end{aligned}$$

Now, can we reverse the process? That is, can we start with $x^2+8x-48$ and place it in its original factored form $(x+12)(x-4)$? The answer is yes, if we apply the following procedure.

The ac-method

Compare the given polynomial with the standard form ax^2+bx+c, determine the coefficients a, b, and c, then proceed as follows:

1. Multiply the coefficients a and c and determine their product ac. List all the integer pairs whose product equals ac.
2. Circle the pair in the list produced in step 1 whose sum equals b, the coefficient of the middle term of ax^2+bx+c.
3. Replace the middle term bx with a sum of like terms using the circled pair from step 2.
4. Factor by grouping.
5. Check the result using the **FOIL** shortcut.

Let's follow the steps of the ac-method to factor $x^2+8x-48$.

Example 6.3.3

Factor: $x^2+8x-48$.

Solution

Compare $x^2+8x-48$ with ax^2+bx+c and identify $a=1$, $b=8$, and $c=-48$. Note that the leading coefficient is $a=1$. Calculate ac. Note that $ac=(1)(-48)$, so $ac=-48$. List all integer pairs whose product is $ac=-48$.

$$\begin{array}{ll}1,-48 & -1,48\\ 2,-24 & -2,24\\ 3,-16 & -3,16\\ 4,-12 & -4,12\\ 6,-8 & -6,8\end{array}$$

Circle the ordered pair whose sum is $b=8$.

$$\begin{array}{ll}1,-48 & -1,48\\ 2,-24 & -2,24\\ 3,-16 & -3,16\\ 4,-12 & -4,12\\ 6,-8 & -6,8\end{array}$$

Replace the middle term $8x$ with a sum of like terms using the circled pair whose sum is 8.

$$x^2+8x-48=x^2-4x+12x-48$$

Factor by grouping.

$$\begin{aligned}x^2+8x-48 &= x(x-4)+12(x-4)\\ &= (x+12)(x-4)\end{aligned}$$

Use the **FOIL** shortcut to mentally check your answer. To determine the product $(x+12)(x-4)$, use these steps:

- Multiply the terms in the "First" positions: x^2.
- Multiply the terms in the "Outer" and "Inner" positions and combine the results mentally: $-4x+12x=8x$.
- Multiply the terms in the "Last" positions: -48.

That is:

$$(x+12)(x-4)=\begin{array}{ccccccc}F & & O & & I & & L\\ x^2 & - & 4x & + & 12x & - & 48\end{array}$$

Combining like terms, $(x+12)(x-4)=x^2+8x-48$, which is the original trinomial, so our solution checks. Note that if you combine the “Outer” and “Inner” products mentally, the check goes even faster.

Exercise 6.3.3

Factor: $x^2+11x+28$

Answer

$(x+4)(x+7)$

Some readers might ask “Is it a coincidence that the circled pair $-4, 12$ seemed to ‘drop in place’ in the resulting factorization $(x+12)(x-4)$?” Before we answer that question, let’s try another example.

Example 6.3.4

Factor: $x^2-9x-36$.

Solution

Compare $x^2-9x-36$ with ax^2+bx+c and note that $a=1$, $b=-9$, and $c=-36$. Calculate $ac=(1)(-36)$, so $ac=-36$.

At this point, some readers might ask “What if I start listing the ordered pairs and I see the pair I need? Do I need to continue listing the remaining pairs?

The answer is “No.” In this case, we start listing the integer pairs whose product is $ac=-36$, but are mindful that we need an integer pair whose sum is $b=-9$. The integer pair 3 and -12 has a product equaling $ac=-36$ and a sum equaling $b=-9$.

$$\begin{array}{c} 1,-36 \\ 2,-18 \\ 3,-12 \end{array}$$

Note how we ceased listing ordered pairs the moment we found the pair we needed. Next, replace the middle term $-9x$ with a sum of like terms using the circled pair.

$$x^2-9x-36=x^2+3x-12x-36$$

Factor by grouping.

$$\begin{aligned} x^2-9x-36 &= x(x+3)-12(x+3) \\ &= (x-12)(x+3) \end{aligned}$$

Use the **FOIL** shortcut to check your answer.

$$(x+3)(x-12)=\begin{array}{ccccccc} F & & O & & I & & L \\ x^2 & - & 12x & + & 3x & - & 36 \end{array}$$

Combining like terms, $(x+3)(x-12)=x^2-9x-36$, the original trinomial. Our solution checks.

Exercise 6.3.4

Factor: $x2+10x-24$.

Answer

$(x+12)(x-2)$

Speeding Things Up a Bit

Readers might again ask “Is it a coincidence that the circled pair $3, -12$ seemed to ‘drop in place’ in the resulting factoriztion $(x-12)(x+3)$?” The answer is “No,” it is not a coincidence. Provided the leading coefficient of the trinomial ax^2+bx+c is $a=1$, you can always “drop in place” the circled pair in order to arrive at the final factorization, skipping the factoring by grouping.

Some readers might also be asking “Do I really have to list any of those ordered pairs if I already recognize the pair I need?” The answer is “No!” If you see the pair you need, drop it in place.

Example 6.3.5

Factor: $x^2 - 5x - 24$.

Solution

Compare $x^2 - 5x - 24$ with $ax^2 + bx + c$ and note that $a = 1$, $b = -5$, and $c = -24$. Calculate $ac = (1)(-24)$, so $ac = -24$. Now can you think of an integer pair whose product is $ac = -24$ and whose sum is $b = -5$? For some, the required pair just pops into their head: -8 and 3. The product of these two integers is -24 and their sum is -5.“Drop” this pair in place and you are done.

$$x^2 - 5x - 24 = (x-8)(x+3)$$

Use the **FOIL** shortcut to check your answer.

$$(x-8)(x+3) = \begin{array}{ccccccc} F & & O & & I & & L \\ x^2 & + & 3x & - & 8x & - & 24 \end{array}$$

Combining like terms, $(x-8)(x+3) = x^2 - 5x - 24$, the original trinomial. Our solution checks.

Exercise 6.3.5

Factor: $x^2 - 12x + 35$

Answer

$(x-7)(x-5)$

The “Drop in Place” technique of Example $6.3.5$ allows us to revise the ac method a bit.

Revised ac-method

Compare the given polynomial with the standard form $ax^2 + bx + c$, determine the coefficients a, b, and c, then determine a pair of integers whose product equals ac and whose sum equals b. You then have two options:

1. Write the middle term as a product of like terms using the ordered pair whose product is ac and whose sum is b. Complete the factorization process by factoring by grouping.
2. (**Only works if** $a = 1$.) Simply “drop in place” the ordered pair whose product is ac and whose sum is b to complete the factorization process. Note: We'll learn about factoring $ax^2 + bx + c$ when $a \neq 1$ in section 6.4 why this “drop in place” choice does not work when $a \neq 1$

Readers are strongly encouraged to check their factorization by determining the product using the **FOIL** method. If this produces the original trinomial, the factorization is correct.

Nonlinear Equations Revisited

The ability to factor trinomials of the form $ax^2 + bx + c$, where $a = 1$, increases the number of nonlinear equations we are now able to solve.

Example 6.3.6

Solve the equation $x^2 = 2x + 3$ both algebraically and graphically, then compare your answers.

Solution

Because there is a power of x larger than one, the equation is nonlinear. Make one side zero.

$$\begin{aligned} x^2 &= 2x + 3 && \text{Original equation.} \\ x^2 - 2x &= 3 && \text{Subtract } 2x \text{ from both sides.} \\ x^2 - 2x - 3 &= 0 && \text{Subtract } 3 \text{ from both sides.} \end{aligned}$$

Compare x^2-2x-3 with ax^2+bx+c and note that $a=1$, $b=-2$ and $c=-3$. We need an integer pair whose product is $ac=-3$ and whose sum is $b=-2$. The integer pair 1 and -3 comes to mind. "Drop" these in place to factor.

$$(x+1)(x-3)=0 \quad \text{Factor.}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$\begin{aligned} x+1&=0 \\ x&=-1 \end{aligned}$$

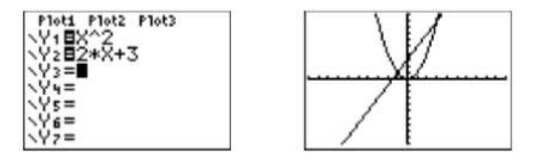
or

$$\begin{aligned} x-3&=0 \\ x&=3 \end{aligned}$$

Thus, the solutions of $x^2=2x+3$ are $x=-1$ and $x=3$.

Graphical solution:

Load each side of the equation $x^2=2x+3$ into the **Y= menu** of your graphing calculator, $y=x^2$ in $\mathtt{Y1}$, $y=2x+3$ in $\mathtt{Y2}$ (see Figure $6.3.1$). Select **6:ZStandard** from the **ZOOM** menu to produce the image at the right in Figure $6.3.1$.

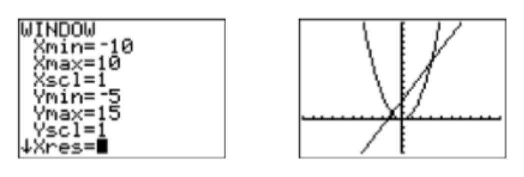


```
Plot1 Plot2 Plot3
\Y1=X^2
\Y2=2*X+3
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure $6.3.1$: Sketch $y=x^2$ and $y=2x+3$.

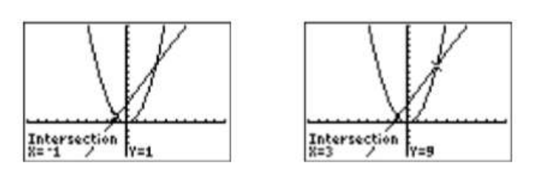
One of the intersection points is visible on the left, but the second point of intersection is very near the top of the screen at the right (see Figure $6.3.1$). Let's extend the top of the screen a bit. Press the **WINDOW** button and make adjustments to $\mathtt{Ymin}$ and $\mathtt{Ymax}$ (see Figure $6.3.2$), then press the **GRAPH** button to adopt the changes.

Note that both points of intersection are now visible in the viewing window (see Figure $6.3.2$). To find the coordinates of the points of intersection, select **5:intersect** from the **CALC** menu. Press the **ENTER** key to accept the "First curve," press **ENTER** again to accept the "Second curve," then press **ENTER** again to accept the current position of the cursor as your guess. The result is shown in the image on the left in Figure $6.3.3$. Repeat the process to find the second point of intersection, only when it comes time to enter your "Guess," use the right-arrow key to move the cursor closer to the second point of intersection than the first.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-5
 Ymax=15
 Yscl=1
↓Xres=
```

Figure $6.3.2$: Adjusting the view.



```
Intersection
X=-1          Y=1
```

```
Intersection
X=3           Y=9
```

Figure $6.3.3$: Use **5:intersect** from the **CALC** menu to find points of intersection.

Reporting the solution on your homework:

Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

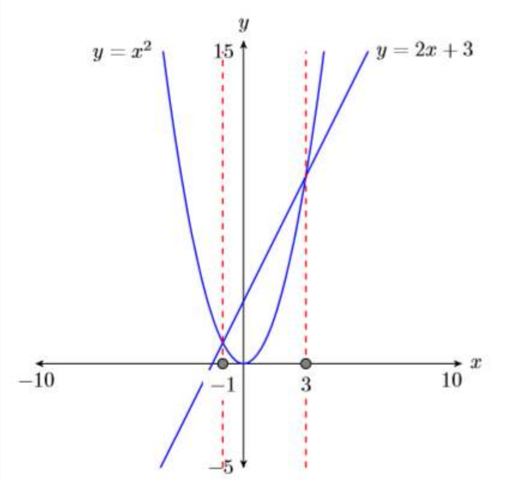


Figure $6.3.4$: Reporting your graphical solution on your homework.

- Label the horizontal and vertical axes with x and y, respectively (see Figure $6.3.4$).
- Place your **WINDOW** parameters at the end of each axis (see Figure $6.3.4$).
- Label each graph with its equation (see Figure $6.3.4$).
- Drop dashed vertical lines through each point of intersection. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 = 2x + 3$ (see Figure $6.3.4$).

Finally, note how the graphical solutions of $x^2 = 2x + 3$, namely $x = -1$ and $x = 3$, match the solutions found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, substituting -1 for x and 3 for x.

$$\begin{aligned} x^2 &= 2x + 3 \\ (-1)^2 &= 2(-1) + 3 \\ 1 &= -2 + 3 \end{aligned}$$

and

$$\begin{aligned} x^2 &= 2x + 3 \\ (3)^2 &= 2(3) + 3 \\ 9 &= 6 + 3 \end{aligned}$$

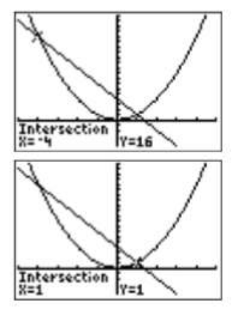
Because the last two statements are true statements, the solutions $x = -1$ and $x = 3$ check in the original equation $x^2 = 2x + 3$.

Exercise $6.3.6$

Solve the equation $x^2 = -3x + 4$ both algebraically and graphically, then compare your answers.

Answer

-4, 1



Intersection
X=-4 Y=16

Intersection
X=1 Y=1

Example 6.3.7

Solve the equation $x^2 - 4x - 96 = 0$ both algebraically and graphically, then compare your answers.

Solution

Because there is a power of x larger than one, the equation $x^2 - 4x - 96 = 0$ is nonlinear. We already have one side zero, so we can proceed with the factoring. Begin listing integer pairs whose product is $ac = -96$, mindful of the fact that we need a pair whose sum is $b = -4$.

$$\begin{array}{c} 1, -96 \\ 2, -48 \\ 3, -32 \\ 4, -24 \\ 6, -16 \\ \color{red}{8, -12} \end{array}$$

Note that we stopped the listing process as soon as we encountered a pair whose sum was $b = -4$. “Drop” this pair in place to factor the trinomial.

$$\begin{aligned} x^2 - 4x - 96 &= 0 && \color{red}{\text{Original equation.}} \\ (x+8)(x-12) &= 0 && \color{red}{\text{Factor.}} \end{aligned}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$\begin{aligned} x + 8 &= 0 \\ x &= -8 \end{aligned}$$

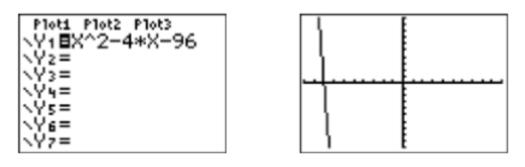
or

$$\begin{aligned} x - 12 &= 0 \\ x &= 12 \end{aligned}$$

Thus, the solutions of $x^2 - 4x - 96 = 0$ are $x = -8$ and $x = 12$.

Graphical solution:

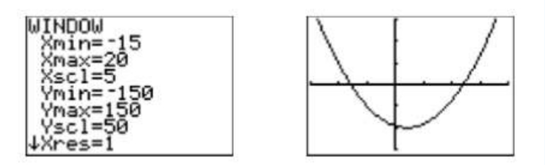
Load the equation $y = x^2 - 4x - 96$ in $\mathtt{Y1}$ in the **Y= menu** of your graphing calculator (see Figure 6.3.5). Select **6:ZStandard** from the **ZOOM** menu to produce the image at the right in Figure 6.3.5.



```
Plot1  Plot2  Plot3
\Y1■X^2-4*X-96
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 6.3.5: Sketch the graph of $y = x^2 - 4x - 96$.

When the degree of a polynomial is two, we’re used to seeing some sort of parabola. In Figure 6.3.5, we saw the graph go down and off the screen, but we did not see it turn and come back up. Let’s adjust the **WINDOW** parameters so that the vertex (turning point) of the parabola and both x-intercepts are visible in the viewing window. After some experimentation, the settings shown in Figure 6.3.6 reveal the vertex and the x-intercepts. Press the **GRAPH** button to produce the image at the right in Figure 6.3.6.



```
WINDOW
 Xmin=-15
 Xmax=20
 Xscl=5
 Ymin=-150
 Ymax=150
 Yscl=50
↓Xres=1
```

Figure 6.3.6: Adjusting the view.

Note that both x-intercepts of the parabola are now visible in the viewing window (see Figure 6.3.6). To find the coordinates of the x-intercepts, select **2:zero** from the **CALC** menu. Use the left- and right-arrow keys to move the cursor to the left of the first x-intercept, then press **ENTER** to mark the “Left bound.” Next, move the cursor to the right of the first x-intercept, then press **ENTER** to mark the "Right bound." Press **ENTER** to accept the current position of the cursor as your "Guess." The result is shown in the image on the left in Figure 6.3.7. Repeat the process to find the coordinates of the second x-intercept. The result is show in the image on the right in Figure 6.3.7.

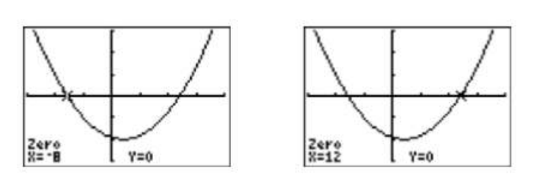


Figure 6.3.7: Use **2:zero** from the **CALC** menu to find the x-intercepts.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 6.3.8).
- Place your WINDOW parameters at the end of each axis (see Figure 6.3.8).
- Label the graph with its equation (see Figure 6.3.8).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 - 4x - 96 = 0$ (see Figure 6.3.8).

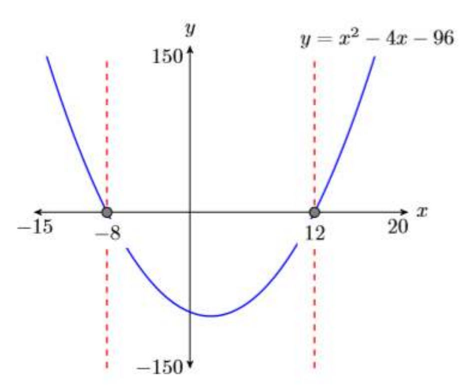


Figure 6.3.8: Reporting your graphical solution on your homework.

Finally, note how the graphical solutions of $x^2 - 4x - 96 = 0$, namely $x = -8$ and $x = 12$, match the solutions found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, substituting -8 for x and 12 for x.

$$\begin{aligned} x^2 - 4x - 96 &= 0 \\ (-8)^2 - 4(-8) - 96 &= 0 \\ 64 + 32 - 96 &= 0 \end{aligned}$$

and

$$\begin{aligned} x^2 - 4x - 96 &= 0 \\ (12)^2 - 4(12) - 96 &= 0 \\ 144 - 48 - 96 &= 0 \end{aligned}$$

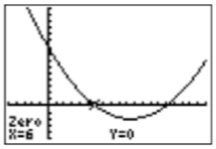
Because the last two statements are true statements, the solutions $x = -8$ and $x = 12$ check in the original equation $x^2 - 4x - 96 = 0$.

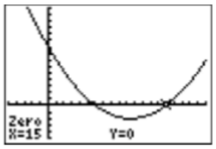
Exercise 6.3.7

Solve the equation $x^2 - 21x + 90 = 0$ both algebraically and graphically, then compare your answers.

Answer

6, 15





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6.4: Factoring ax² + bx + c when a≠1

In this section we continue to factor trinomials of the form ax^2+bx+c. In the last section, all of our examples had $a=1$, and we were able to "Drop in place" our circled integer pair. However, in this section, $a\neq 1$, and we'll soon see that we will not be able to use the "Drop in place" technique. However, readers will be pleased to learn that the ac-method will still apply.

Example 6.4.1

Factor: $2x^2-7x-15$.

Solution

We proceed as follows:

1. Compare $2x^2-7x-15$ with ax^2+bx+c and identify $a=2$, $b=-7$, and $c=-15$. Note that the leading coefficient is $a=2$, so this case is different from all of the cases discussed in Section 6.3.
2. Calculate ac. Note that $ac=(2)(-15)$, so $ac=-30$.
3. List all integer pairs whose product is $ac=-30$.

$$\begin{array}{ll} 1,-30 & -1,30 \\ 2,-15 & -2,15 \\ 3,-10 & -3,10 \\ 5,-6 & -5,6 \end{array}$$

4. Circle the ordered pair whose sum is $b=-7$.

$$\begin{array}{ll} 1,-30 & -1,30 \\ 2,-15 & -2,15 \\ 3,-10 & -3,10 \\ 5,-6 & -5,6 \end{array}$$

5. Note that if we "drop in place" our circled ordered pair, $(x+3)(x-10)\neq 2x^2-7x-15$. Right off the bat, the product of the terms in the "First" position does not equal $2x^2$. Instead, we break up the middle term of $2x^2-7x-15$ into a sum of like terms using our circled pair of integers 3 and -10.

$$2x^2-7x-15=2x^2+3x-10x-15$$

Now we factor by grouping. Factor x out of the first two terms and -5 out of the second two terms.

$$=x(2x+3)-5(2x+3)$$

Now we can factor out (2x + 3).

$$=(x-5)(2x+3)$$

6. Use the FOIL shortcut to mentally check your answer. To multiply $(x-5)(2x+3)$, use these steps:
   - Multiply the terms in the "First" positions: $2x^2$.
   - Multiply the terms in the "Outer" and "Inner" positions and combine the results mentally: $3x-10x=-7x$.
   - Multiply the terms in the "Last" positions: -15. That is:

$$(x-5)(2x+3)=\begin{array}{ccccccc} F & & O & & I & & L \\ 2x^2 & + & 3x & - & 10x & - & 15 \end{array}$$

Combining like terms, $(x-5)(2x+3)=2x^2-7x-15$, which is the original trinomial, so our solution checks. Note that if you combine the "Outer" and "Inner" products mentally, the check goes even faster.

Exercise 6.4.1

Factor: $3x^2+13x+14$

Answer

$(x+2)(3x+7)$

Speeding Things Up a Bit

Some readers might already be asking “Do I really have to list all of those ordered pairs if I already see the pair I need?” The answer is “No!” If you see the pair you need, use it to break up the middle term of the trinomial as a sum of like terms.

Example 6.4.2

Factor: $3x^2-7x-6$.

Solution

Compare $3x^2-7x-6$ with ax^2+bx+c and note that $a=3$, $b=-7$, and $c=-6$. Calculate $ac=(3)(-6)$, so $ac=-18$. Now can you think of an integer pair whose product is $ac=-18$ and whose sum is $b=-7$? For some, the pair just pops into their head: 2 and -9. Break up the middle term into a sum of like terms using the pair 2 and -9.

$$\begin{aligned} 3x^2-7x-6 &= 3x^2+2x-9x-6 && -7x=2x-9x \\ &= x(3x+2)-3(3x+2) && \text{Factor by grouping.} \\ &= (x-3)(3x+2) && \text{Factor out}(3x+2). \end{aligned}$$

Use the FOIL shortcut to check your answer.

$$(x-3)(3x+2)=\begin{array}{ccccccc} F & & O & & I & & L \\ 3x^2 & + & 2x & - & 9x & - & 6 \end{array}$$

Combining like terms, $(x-3)(3x+2)=3x^2-7x-6$, the original trinomial. Our solution checks. Note that if you combine the “Outer” and “Inner” products mentally, the check goes even faster.

Exercise 6.4.2

Factor: $2x^2-9x+10$

Answer

$(x-2)(2x-5)$

On the other hand, some readers might be saying “Well, the needed ordered pair is not popping into my head. Do I have a way of cutting down the work?” The answer is “Yes!” As you are listing the ordered pairs whose product equals ac, be mindful that you need the ordered pair whose sum is b. If you stumble across the needed pair, stop the listing process and “drop” your ordered pair in place.

Example 6.4.3

Factor: $3x^2-33x+54$.

Solution

Compare $3x^2-33x+54$ with ax^2+bx+c and note that $a=3$, $b=-33$, and $c=54$. Calculate $ac=(3)(54)$, so $ac=162$. Ouch! That’s a big number! However, start listing the integer pairs whose product is $ac=162$, but be mindful that you need an integer pair whose sum is $b=-33$.

$$\begin{array}{ll} 1,162 & \\ 2,81 & \\ 3,54 & \\ 6,27 & -6,-27 \end{array}$$

As soon as we wrote down the pair 6 and 27, our mind said “the sum of 6 and 7 is 33.” However, we need the sum to equal $b=-33$, so we boxed -6 and -27 instead. Next, we break up the middle term into a sum of like terms using our circled pair.

$$\begin{aligned} 3x^2 - 33x - 54 &= 3x^2 - 6x - 27x - 54 \quad -33x = -6x - 27x \\ &= 3x(x-2) - 27(x-2) \quad \text{Factor by grouping.} \\ &= (3x-27)(x-2) \quad \text{Factor out}(x-2). \end{aligned}$$

Oh-oh! Now we realize we can factor 3 out of each term in the first factor!

$$= 3(x-9)(x-2)$$

We missed taking out the GCF! Let's try again, only this time let's do what we are always supposed to do in the first step: Factor out the GCF.

$$3x^2 - 33x + 54 = 3(x^2 - 11x + 18)$$

Comparing $x^2 - 11x + 18$ with $ax^2 + bx + c$, we see that $a = 1$, $b = -11$, and $c = 18$. We need an integer pair whose product is $ac = 18$ and whose sum is $b = -11$. Note that these numbers are considerably smaller than the numbers we had to deal with when we forgot to first factor out the GCF. Because the numbers are smaller, the integer pair -9 and -2 easily comes to mind. Furthermore, because $a = 1$, we can factor $x^2 - 11x + 18$ by simply dropping the integer pair -9 and -2 in place.

$$3(x2 - 11x + 18) = 3(x-9)(x-2)$$

A far simpler solution!

Exercise 6.4.3

Factor: $5x^2 - 35x - 40$

Answer

$5(x-8)(x+1)$

In Example 6.4.3, we saw how much more difficult we made the problem by forgetting to first factor out the greatest common factor (GCF). Let's try not to make that mistake again.

First rule of factoring

The first step in factoring any polynomial is to factor out the greatest common factor.

Example 6.4.4

Factor: $30x^3 - 21x^2 - 18x$.

Solution

Note that the GCF of $30x^3$, $21x^2$, and $18x$ is $3x$. Factor out this GCF.

$$\begin{aligned} 30x^3 - 21x^2 - 18x &= 3x \cdot 10x^2 - 3x \cdot 7x - 3x \cdot 6 \\ &= 3x(10x^2 - 7x - 6) \end{aligned}$$

Next, compare $10x^2 - 7x - 6$ with $ax^2 + bx + c$ and note that $a = 10$, $b = -7$, and $c = -6$. Start listing the integer pairs whose product is $ac = -60$, but be mindful that you need an integer pair whose sum is $b = -7$.

$$\begin{gathered} 1, -60 \\ 2, -30 \\ 3, -20 \\ 4, -15 \\ 5, -12 \end{gathered}$$

Break up the middle term into a sum of like terms using our circled pair.

$$\begin{aligned} 3x(10x^2 - 7x - 6) &= 3x(10x^2 + 5x - 12x - 18) \quad -7x = 5x - 12x \\ &= 3x[5x(2x+1) - 6(2x+1)] \quad \text{Factor by grouping.} \\ &= 3x(5x-6)(2x+1) \quad \text{Factor out}(2x+1). \end{aligned}$$

Hence, $30x^3 - 21x^2 - 18x = 3x(5x-6)(2x+1)$.

Check: First, use the FOIL shortcut to multiply the two binomial factors, then distribute the monomial factor.

$$\begin{aligned} 3x(5x-6)(2x+1) &= 3x(10x^2-7x-6) \quad \text{Apply the FOIL shortcut} \\ &= 3x^3 - 21x^2 - 18x \quad \text{Distribute the } 3x. \end{aligned}$$

Because this is the original polynomial, the solution checks.

Exercise $6.4.4$

Factor: $12x^4 + 2x^3 - 30x^2$

Answer

$2x^2(3x+5)(2x-3)$

Nonlinear Equations Revisited

Let's use the factoring technique of this chapter to solve some nonlinear equations.

Example $6.4.5$

Solve the equation $2x^2 = 13x - 20$ both algebraically and graphically, then compare your answers.

Solution

Because there is a power of x larger than one, the equation is nonlinear. Make one side equal to zero.

$$\begin{aligned} 2x^2 &= 13x - 20 \quad \text{Original equation} \\ 2x^2 - 13x + 20 &= 0 \quad \text{Make one side zero.} \end{aligned}$$

Compare $2x^2 - 13x + 20$ with $ax^2 + bx + c$ and note that $a = 2$, $b = -13$ and $c = 20$. We need an integer pair whose product is $ac = 40$ and whose sum is $b = -13$. The integer pair -5 and -8 comes to mind. Write the middle term as a sum of like terms using this pair.

$$\begin{aligned} 2x^2 - 5x - 8x + 20 &= 0 \quad -13x = -5x - 8x \\ x(2x-5)(2x-5) &= 0 \quad \text{Factor by grouping.} \\ (x-4)(2x-5) &= 0 \quad \text{Factor out } 2x-5 \end{aligned}$$

We have a product that equals zero. Use the zero product property to complete the solution.

$$\begin{aligned} x - 4 &= 0 \\ x &= 4 \end{aligned}$$

or

$$\begin{aligned} 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Thus, the solutions of $2x^2 = 13x - 20$ are $x = 4$ and $x = 5/2$.

Graphical solution:

Load each side of the equation $2x^2 = 13x - 20$ into the **Y= menu** of your graphing calculator, $y = 2x^2$ in $\mathtt{Y1}$, $y = 13x - 20$ in $\mathtt{Y2}$ (see Figure $6.4.1$). Select **6:ZStandard** from the **ZOOM** menu to produce the image on the left in Figure 6.18. However, even after adjusting the **WINDOW** parameters ($\mathtt{Xmin} = -10$, $\mathtt{Xmax} = 10$, $\mathtt{Ymin} = -10$, and $\mathtt{Ymax} = 60$), the image resulting from pushing the **GRAPH** button (see the image on the right in Figure $6.4.1$) does not clearly show the two points of intersection.

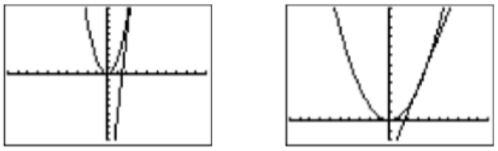
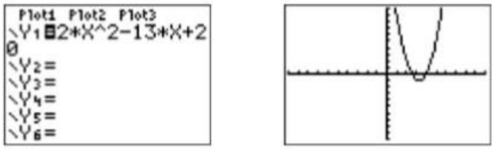


Figure 6.4.1: Sketch $y = 2x^2$ and $y = 13x - 20$.

Let's switch our strategy and work with the equation $2x2 - 13x + 20 = 0$ instead. Load $y = 2x^2 - 13x + 20$ into `Y1` in the **Y= menu**, then select **6:ZStandard** from the **ZOOM** menu to produce the image at the right in Figure 6.4.2.



```
Plot1 Plot2 Plot3
\Y1■2*X^2-13*X+2
0
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Figure 6.4.2: Sketch $y = 2x^2 - 13x + 20$.

To find the solutions of $2x^2 - 13x + 20 = 0$, we must identify the x-intercepts of the graph in Figure 6.4.1. Select **2:zero** from the **CALC** menu, then move the **left- and right=arrows** to move the cursor to the left of the first x-intercept. Press **ENTER** to mark the "Left bound," then move the cursor to the right of the x-intercept and press **ENTER** to mark the "Right bound." Finally, press **ENTER** to use the current position of the cursor for your "Guess." The result is shown in the image on the left in Figure 6.4.3. Repeat the process to find the rightmost x-intercept. The result is shown in the image on the right in Figure 6.4.3.

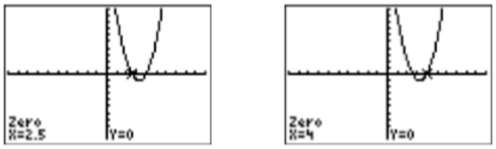


Figure 6.4.3: Use **2:zero** from the **CALC** menu to find the x-intercepts.

Reporting the solution on your homework:

Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 6.4.4).
- Place your **WINDOW** parameters at the end of each axis (see Figure 6.4.4).
- Label the graph with its equation (see Figure 6.4.4).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $2x^2 - 13x + 20 = 0$ (see Figure 6.4.4).

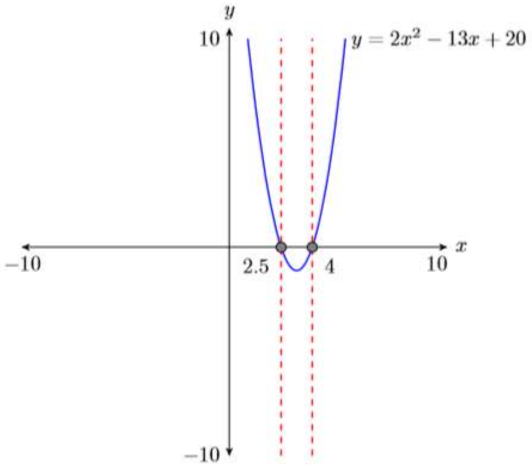


Figure 6.4.4: Reporting your graphical solution on your homework.

Finally, note how the graphical solutions of $2x^2-13x+20=0$, namely $x=2.5$ and $x=4$, match the solutions $x=5/2$ and $x=4$ found using the algebraic method. This is solid evidence that both methods of solution are correct. However, it doesn't hurt to check the final answers in the original equation, substituting $5/2$ for x and 4 for x.

$$\begin{aligned}2x^2&=13x-20\\2\left(\frac{5}{2}\right)^2&=13\left(\frac{5}{2}\right)-20\\2\left(\frac{25}{4}\right)&=13\left(\frac{5}{2}\right)-20\\\frac{25}{2}&=\frac{65}{2}-\frac{40}{2}\end{aligned}$$

and

$$\begin{aligned}2x^2&=13x-20\\2(4)^2&=13(4)-20\\2(16)&=13(4)-20\\32&=52-20\end{aligned}$$

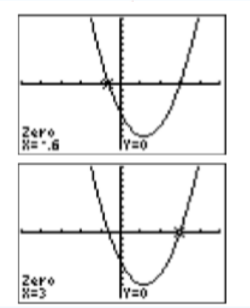
Because the last two statements are true statements, the solutions $x=5/2$ and $x=4$ check in the original equation $2x^2=13x-20$.

Exercise 6.4.5

Solve the equation $5x^2=12x+9$ both algebraically and graphically, then compare your answers.

Answer

$-3/5$, 3



Example 6.4.6

Solve the equation $2x^3+x^2=28x$ both algebraically and graphically, then compare your answers.

Solution

Because there is a power of x larger than one, the equation is nonlinear. Make one side equal to zero.

$$\begin{aligned}2x^3+x^2&=28x &&\text{Original equation}\\2x^3+x^2-28x&=0 &&\text{Make one side zero.}\end{aligned}$$

Note that the GCF of $2x^3$, x^2, and $28x$ is x. Factor out x.

$$x(2x^2+x-28)=0 \quad \text{Factor out the GCF}$$

Compare $2x^2+x-28$ with ax^2+bx+c and note that $a=2$, $b=1$ and $c=-28$. We need an integer pair whose product is $ac=-56$ and whose sum is $b=1$. The integer pair -7 and 8 comes to mind. Write the middle term as a sum of like terms using this pair.

$$\begin{aligned}x(2x^2-7x+8x-28)&=0 &&x=-7x+8x\\x[x(2x-7)+4(2x-7)]&=0 &&\text{Factor by grouping.}\\x(x+4)(2x-7)&=0 &&\text{Factor out }2x-7.\end{aligned}$$

We have a product of three factors that equals zero. By the zero product property, at least one of the factors must equal zero.

$$x=0$$

or

$$\begin{aligned} x+4&=0\\ x&=-4 \end{aligned}$$

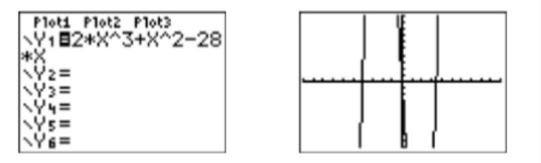
or

$$\begin{aligned} 2x-7&=0\\ 2x&=7\\ x&=\frac{7}{2} \end{aligned}$$

Thus, the solutions of $2x^3+x^2=28x$ are $x=0$, $x=-4$, and $x=7/2$.

Graphical solution:

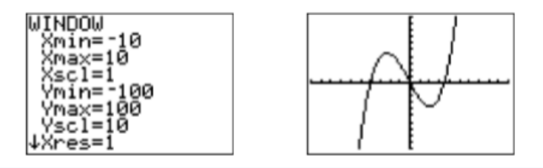
Rather than working with $2x^3+x^2=28x$, graphing each side separately and finding where the graphs intersect, we will work instead with $2x^3+x^2-28x=0$, locating where the graph of $y=2x^3+x^2-28x$ crosses the x-axis. Load $y=2x^3+x^2-28x$ into $\mathbb{Y}1$ in the **Y= menu**, then select **6:ZStandard** from the **ZOOM** menu to produce the image at the right in Figure $6.4.5$.



```
Plot1 Plot2 Plot3
\Y1■2*X^3+X^2-28
*X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Figure $6.4.5$: Sketch $y=2x^3+x^2-28x$.

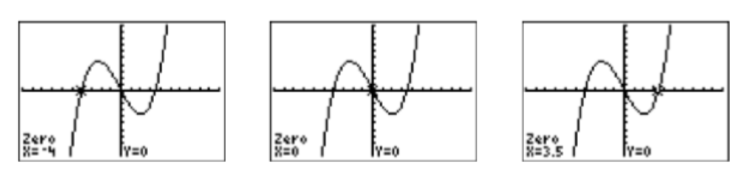
In the image at the right in Figure $6.4.6$, we saw the graph rise from the bottom of the screen, leave the top of the screen, return and leave via the bottom of the screen, and then finally return and leave via the top of the screen. Clearly, there are at least two turning points to the graph that are not visible in the current viewing window. Set the **WINDOW** settings as shown in the image on the left in Figure $6.4.6$, then push the **GRAPH** button to produce the image on the right in Figure $6.4.6$. Note that this window now shows the x-intercepts as well as the turning points of the graph of the polynomial.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-100
 Ymax=100
 Yscl=10
↓Xres=1
```

Figure $6.4.6$: Adjusting the view so that the turning points of the polynomial are visible.

To find the solutions of $2x^3+x2-28x=0$, we must identify the x-intercepts of the graph in Figure $6.4.6$. Select **2:zero** from the **CALC** menu, then use the left- and right-arrow keys to move the cursor to the left of the first x-intercept. Press **ENTER** to mark the “Left bound,” then move the cursor to the right of the x-intercept and press **ENTER** to mark the “Right bound.” Finally, press **ENTER** to use the current position of the cursor for your “Guess.” The result is shown in the first image on the left in Figure $6.4.7$. Repeat the process to find the remaining x-intercepts. The results are shown in the next two images in Figure $6.4.7$.



```
Zero
X=-4      Y=0
```

```
Zero
X=0       Y=0
```

```
Zero
X=3.5     Y=0
```

Figure $6.4.7$: Use **2:zero** from the **CALC** menu to find the x-intercepts.

Reporting the solution on your homework:

Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure $6.4.8$).
- Place your **WINDOW** parameters at the end of each axis (see Figure $6.4.8$).
- Label the graph with its equation (see Figure $6.4.8$).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $2x^3+x^2-28x=0$ (see Figure $6.4.8$).

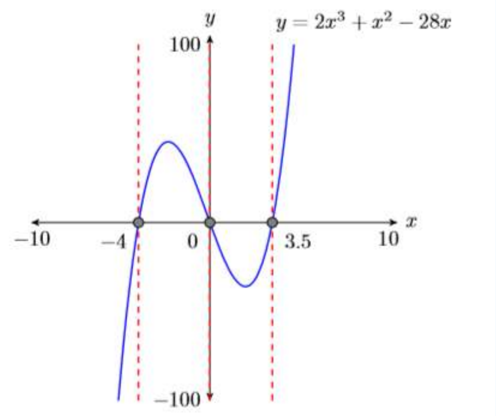


Figure $6.4.8$: Reporting your graphical solution on your homework.

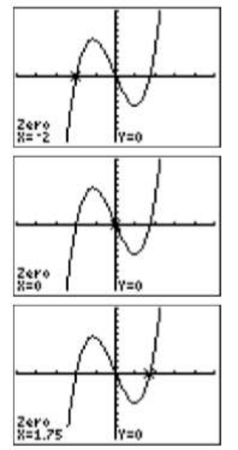
Finally, note how the graphical solutions of $2x^3+x^2-28x=0$, namely $x=-4$, $x=0$, and $x=3.5$, match the solutions $x=-4$, $x=0$, and $x=7/2$ found using the algebraic method.

Exercise $6.4.6$

Solve the equation $4x^3=-x^2+14x$ both algebraically and graphically, then compare your answers.

Answer

-2, 0, $7/4$



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6.5: Factoring Special Forms

In this section we revisit two special product forms that we learned in Chapter 5, the first of which was *squaring a binomial*.

Squaring a binomial

Here are two earlier rules for squaring a binomial.

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$

Perfect Square Trinomials

To square a binomial such as $(a+b)^2$, proceed as follows:

1. Square the first term: a
2. Multiply the first and second term, then double: $2ab$
3. Square the last term: b^2

Example 6.5.1

Expand: $(2x+3y)^2$

Solution

Using the pattern $(a+b)^2 = a^2 + 2ab + b^2$, we can expand $(2x+3y)^2$ as follows:

$$\begin{aligned}(2x+3y)^2 &= (2x)^2 + 2(2x)(3y) + (3y)^2 \\ &= 4x^2 + 6xy + 9y^2\end{aligned}$$

Note how we square the first and second terms, then produce the middle term of our answer by multiplying the first and second terms and doubling.

Exercise 6.5.1

Expand: $(5a+2b)^2$

Answer

$25a^2 + 20ab + 4b^2$

Example 6.5.2

Expand: $(3u^2 - 5v2)^2$

Solution

Using the pattern $(a-b)^2 = a^2 - 2ab + b^2$, we can expand $(3u^2 - 5v^2)^2$ as follows:

$$\begin{aligned}(3u^2 - 5v^2)^2 &= (3u^2)^2 - 2(3u^2)(5v^2) + (5v^2)^2 \\ &= 9u^4 - 30u^2v^2 + 25v^4\end{aligned}$$

Note that the sign of the middle term is negative this time. The first and last terms are still positive because we are squaring.

Exercise 6.5.2

Expand: $(2s^3 - 7t)^2$

Answer

$4s^6 - 28s^3t + 49t^2$

Once you've squared a few binomials, it's time to do all of the work in your head.

i. Square the first term.
ii. Multiply the first and second term and double the result.
iii. Square the second term.

Example $6.5.3$

Expand each of the following:

a. $(2y-3)^2$
b. $(4a-3b)^2$
c. $(x3+5)^2$

Solution

Using the pattern $(a\pm b)^2 = a^2 \pm 2ab + b^2$, we expand each binomially mentally, writing down the answer without any intermediate steps.

a. $(2y-3)^2 = 4y^2-12y+9$
b. $(4a-3b)^2 = 16a^2-24ab+9b^2$
c. $(x^3+5)^2 = x^6+10x^3+25$

Exercise $6.5.3$

Expand: $(5x^4-3)^2$

Answer

$25x^8-30x^4+9$

Now, because factoring is "unmultiplying," it should be a simple matter to reverse the process of Example $6.5.3$.

Example $6.5.4$

Factor each of the following trinomials:

a. $4y^2-12y+9$
b. $16a^2-24ab+9b^2$
c. x^6+10x^3+25

Solution

Because of the work already done in Example $6.5.3$, it is a simple task to factor each of these trinomials.

a. $4y^2-12y+9=(2y-3)^2$
b. $16a^2-24ab+9b^2=(4a-3b)^2$
c. $x^6+10x^3+25=(x^3+5)$

Exercise $6.5.4$

Factor: $25x^8-30x^4+9$

Answer

$(5x^4-3)^2$

Each of the trinomials in Example $6.5.4$ is an example of a perfect square trinomial.

Perfect square trinomial

If a trinomial $a^2+2ab+b^2$ is the square of a binomial, as in $(a+b)^2$, then the trinomial is called a perfect square trinomial.

So, how does one recognize a perfect square trinomial? If the first and last terms of a trinomial are perfect squares, then you should suspect that you may be dealing with a perfect square trinomial. However, you also have to have the correct middle term in order to have a perfect square trinomial.

Table $6.5.1$: List of Squares

n	n^2
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441
22	484
23	529
24	576
25	625

Example $6.5.5$

Factor each of the following trinomials:

a. $9x^2 - 42x + 49$
b. $49a^2 + 70ab + 25b^2$
c. $4x^2 - 37x + 9$

Solution

Note that the first and last terms of each trinomial are perfect squares.

a. In the trinomial $9x^2 - 42x + 49$, note that $(3x)^2 = 9x^2$ and $72 = 49$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $9x^2 - 42x + 49$ factors as follows:

$$9x^2 - 42x + 49 \stackrel{?}{=} (3x - 7)^2$$

However, we must check to see if the middle term is correct. Multiply $3x$ and 7, then double: $2(3x)(7) = 42x$. Thus, the middle term is correct and therefore

$$9x^2 - 42x + 49 = (3x - 7)^2$$

b. In the trinomial $49a^2 + 70ab + 25b^2$, note that $(7a)^2 = 49a^2$ and $(5b)^2 = 25b^2$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $49a^2 + 70ab + 25b^2$ factors as follows:

$$49a^2 + 70ab + 25b^2 \stackrel{?}{=} (7a + 5b)^2$$

However, we must check to see if the middle term is correct. Multiply $7a$ and $5b$, then double: $2(7a)(5b) = 70ab$. Thus, the middle term is correct and therefore

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2$$

c. In the trinomial $4x^2 - 37x + 9$, note that $(2x)^2 = 4x^2$ and $(3)^2 = 9$. Hence, the first and last terms are perfect squares. Taking the square roots, we suspect that $4x^2 - 37x + 9$ factors as follows:

$$4x^2 - 37x + 9 \stackrel{?}{=} (2x - 3)^2$$

However, we must check to see if the middle term is correct. Multiply $2x$ and 3, then double: $2(2x)(3) = 12x$. However, this is not the middle term of $4x^2 - 37x + 9$, so this factorization is incorrect! We must find another way to factor this trinomial.

Comparing $4x^2 - 37x + 9$ with $ax^2 + bx + c$, we need a pair of integers whose product is $ac = 36$ and whose sum is $b = -37$. The integer pair -1 and -36 comes to mind. Replace the middle term as a sum of like terms using this ordered pair.

$$\begin{aligned} 4x^2 - 37x + 9 &= 4x^2 - x - 36x + 9 && \color{red}{-37x = -x - 36x} \\ &= x(4x - 1) - 9(4x - 1) && \color{red}{\text{Factor by grouping}} \\ &= (x - 9)(4x - 1) && \color{red}{\text{Factor out } 4x - 1} \end{aligned}$$

This example clearly demonstrates how important it is to check the middle term.

Exercise 6.5.5

Factor: $16x^2 + 72x + 81$

Answer

$(4x + 9)^2$

Remember the first rule of factoring!

The first rule of factoring

The first step to perform in any factoring problem is factor out the GCF.

Example 6.5.6

Factor each of the following trinomials:

a. $2x^3y + 12x^2y^2 + 18xy^3$
b. $-4x^5 + 32x^4 - 64x^3$

Solution

Remember, first factor out the GCF.

a. In the trinomial $2x^3y + 12x^2y^2 + 18xy^3$, we note that the GCF of $2x^3y$, $12x^2y^2$, and $18xy^3$ is $2xy$. We first factor out $2xy$.

$$2x^3y+12x^2y^2+18xy^3=2xy(x^2+6xy+9y^2)$$

We now note that the first and last terms of the resulting trinomial factor are perfect squares, so we take their square roots and factors as follows.

$$=2xy(x+3y)^2$$

Of course, the last factorization is correct only if the middle term is correct. Because $2(x)(3y)=6xy$ matches the middle term of $x^2+6xy+9y^2$, we do have a perfect square trinomial and our result is correct.

b. In the trinomial $-4x^5+32x^4-64x^3$, we note that the GCF of $4x^5$, $32x^4$, and $64x^3$ is $4x^3$. We first factor out $4x^3$.

$$-4x^5+32x^4-64x^3=4x^3(-x^2+8x-16)$$

However, the first and third terms of $-x^2+8x-16$ are negative, and thus are not perfect squares. Let's begin again, this time factoring out $-4x^3$.

$$-4x^5+32x^4-64x^3=-4x^3(x^2-8x+16)$$

This time the first and third terms of $x^2-8x+16$ are perfect squares. We take their square roots and write:

$$=-4x^3(x-4)^2$$

Again, this last factorization is correct only if the middle term is correct. Because $2(x)(4)=8x$, we do have a perfect square trinomial and our result is correct.

Exercise 6.5.6

Factor: $-4x^3-24x^2-36x$

Answer

$-4x(x+3)^2$

The Difference of Squares

The second special product form we learned in Chapter 5 was the *difference of squares*.

The difference of squares

Here is the difference of squares rule.

$$(a+b)(a-b)=a^2-b^2$$

If you are multiplying two binomials which have the exact same terms in the "First" positions and the exact same terms in the "Last" positions, but one set is separated by a plus sign while the other set is separated by a minus sign, then multiply as follows:

1. Square the first term: a^2
2. Square the second term: b^2
3. Place a minus sign between the two squares.

Example 6.5.7

Expand each of the following:

a. $(3x+5)(3x-5)$
b. $(a^3-2b^3)(a^3+2b^3)$

Solution

We apply the difference of squares pattern to expand each of the given problems.

a. In $(3x+5)(3x-5)$, we have the exact same terms in the "First" and "Last" positions, with the first set separated by a plus sign and the second set separated by a minus sign.
    i. Square the first term: $(3x)^2=9x^2$

ii. Square the second term: $5^2 = 25$
iii. Place a minus sign between the two squares.

Hence:

$$(3x+5)(3x-5) = 9x^2 - 25$$

b. In $(a^3 - 2b^3)(a^3 + 2b^3)$, we have the exact same terms in the "First" and "Last" positions, with the first set separated by a minus sign and the second set separated by a plus sign.

   i. Square the first term: $(a^3)^2 = a^6$
   ii. Square the second term: $(2b^3)^2 = 4b^6$
   iii. Place a minus sign between the two squares.

Hence:

$$(a^3 - 2b^3)(a^3 + 2b^3) = a^6 - 4b^6$$

Exercise 6.5.7

Expand: $(4x-3y)(4x+3y)$

Answer

$16x^2 - 9y^2$

Because factoring is "unmultiplying," is should be a simple matter to reverse the process of Example 6.5.7.

Example 6.5.8

Factor each of the following:

a. $9x^2 - 25$
b. $a^6 - 4b^6$

Solution

Because of the work already done in Example 6.5.7, it is a simple matter to factor (or "unmultiply") each of these problems.

a. $9x^2 - 25 = (3x+5)(3x-5)$
b. $a^6 - 4b^6 = (a^3 - 2b^3)(a^3 + 2b^3)$

In each case, note how we took the square roots of each term, then separated one set with a plus sign and the other with a minus sign. Because of the commutative property of multiplication, it does not matter which one you make plus and which one you make minus.

Exercise 6.5.8

Factor: $81x^2 - 49$

Answer

$(9x+7)(9x-7)$

Always remember the first rule of factoring.

Example 6.5.9

Factor: $x^3 - 9x$

Solution

In $x^3 - 9x$, the GCF of x^3 and $9x$ is x. Factor out x.

$$x^3 - 9x = x(x^2 - 9)$$

Note that x^2-9 is now the difference of two perfect squares. Take the square roots of x^2 and 9, which are x and 3, then separate one set with a plus sign and the other set with a minus sign.

$$=x(x+3)(x-3)$$

Exercise 6.5.9

Factor: $4x^4-16x^2$

Answer

$4x^2(x+2)(x-2)$

Factoring Completely

Sometimes after one pass at factoring, factors remain that can be factored further. You must continue to factor in this case.

Example 6.5.10

Factor: x^4-16

Solution

In x^4-16, we have the difference of two squares: $(x^2)^2=x^4$ and $4^2=16$. First, we take the square roots, then separate one set with a plus sign and the other set with a minus sign.

$$x^4-16=(x^2+4)(x^2-4)$$

Note that x^2+4 is the sum of two squares and does not factor further. However, x^2-4 is the difference of two squares. Take the square roots, x and 2, then separate one set with a plus sign and the other set with a minus sign.

$$=(x^2+4)(x+2)(x-2)$$

Done. We cannot factor further.

Exercise 6.5.10

Factor: x^4-81

Answer

$(x^2+9)(x+3)(x-3)$

Nonlinear Equations Revisited

Remember, if an equation is nonlinear, the first step is to make one side equal to zero by moving all terms to one side of the equation. Once you’ve completed this important first step, factor and apply the zero product property to find the solutions.

Example 6.5.11

Solve for x: $25x^2=169$

Solution

Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 25x^2&=169 && \text{Original equation.}\\ 25x^2-169&=0 && \text{Subtract 169 from both sides.}\end{aligned}$$

Note that we have two perfect squares separated by a minus sign. This is the difference of squares pattern. Take the square roots, making one term plus and one term minus.

$$(5x+13)(5x-13)=0 \quad \text{Use difference of squares to factor.}$$

Use the zero product property to complete the solution, setting each factor equal to zero and solving the resulting equations.

$$\begin{aligned} 5x+13&=0 \\ x&=-\frac{13}{5} \end{aligned}$$

or

$$\begin{aligned} 5x-13&=0 \\ x&=\frac{13}{5} \end{aligned}$$

Hence, the solutions of $25x^2=169$ are $x=-13/5$ and $x=13/5$. We encourage readers to check each of these solutions.

Exercise 6.5.11

Solve for x: $16x^2=121$

Answer

$-11/4$, $11/4$

Example 6.5.12

Solve for x: $49x^2+81=126x$

Solution

Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 49x^2+81x&=126x && \text{Original equation.} \\ 49x^2-126x+81&=0 && \text{Subtract 126x from both sides.} \end{aligned}$$

Note that the first and last terms of the trinomial are perfect squares. Hence, it make sense to try and factor as a perfect square trinomial, taking the square roots of the first and last terms.

$$(7x-9)^2=0 \quad \text{Factor as a perfect square trinomial.}$$

Of course, be sure to check the middle term. Because $-2(7x)(9)=-126x$, the middle term is correct. Because $(7x-9)^2=(7x-9)(7x-9)$, we can use the zero product property to set each factor equal to zero and solve the resulting equations.

$$\begin{aligned} 7x-9&=0 \\ x&=\frac{9}{7} \end{aligned}$$

or

$$\begin{aligned} 7x-9&=0 \\ x&=\frac{9}{7} \end{aligned}$$

Hence, the only solution of $49x^2+81=126x$ is $x=9/7$. We encourage readers to check this solution.

Note

One can also argue that the only number whose square is zero is the number zero. Hence, one can go directly from $(7x-9)^2=0$ to $7x-9=0$. Hence, the only solution of $49x^2+81=126x$ is $x=9/7$.

Exercise 6.5.12

Solve for x: $25x^2=80x-64$

Answer

$8/5$

Example 6.5.13

Solve for x: $2x^3 + 3x^2 = 50x + 75$

Solution

Make one side equal to zero, factor, then apply the zero product property.

$$\begin{aligned} 2x^3 + 3x^2 &= 50x + 75 \quad \text{Original equation.} \\ 2x^3 + 3x^2 - 50x - 75 &= 0 \quad \text{Make one side zero.} \end{aligned}$$

This is a four-term expression, so we try factoring by grouping. Factor x^2 out of the first two terms, and -25 out of the second two terms.

$$\begin{aligned} x^2(2x+3) - 25(2x+3) &= 0 \quad \text{Factor by grouping} \\ (x^2 - 25)(2x+3) &= 0 \quad \text{Factor out } 2x+3 \end{aligned}$$

Complete the factorization by using the difference of squares to factor $x^2 - 25$.

$$(x+5)(x-5)(2x+3) = 0 \quad \text{Use difference of squares to factor.}$$

Finally, use the zero product property. Set each factor equal to zero and solve for x.

$$\begin{aligned} x+5 &= 0 \\ x &= -5 \end{aligned}$$

or

$$\begin{aligned} x-5 &= 0 \\ x &= 5 \end{aligned}$$

or

$$\begin{aligned} 2x+3 &= 0 \\ x &= -\frac{3}{2} \end{aligned}$$

Hence, the solutions of $2x^3 + 3x^2 = 50x + 75$ are $x = -5$, $x = 5$, and $x = -3/2$. We encourage readers to check each of these solutions.

Exercise 6.5.13

Solve for x: $5x^3 + 36 = x^2 + 180x$

Answer

-6, 6, $1/5$

Let's solve another nonlinear equation, matching the algebraic and graphical solutions.

Example 6.5.14

Solve the equation $x^3 = 4x$, both algebraically and graphically, then compare your answers.

Solution

Note that we have a power of x larger than one, so the equation $x^3 = 4x$ is nonlinear. Make one side zero and factor.

$$\begin{aligned} x^3 &= 4x \quad \text{Original equation.} \\ x^3 - 4x &= 0 \quad \text{Nonlinear. Make one side zero.} \\ x(x^2 - 4) &= 0 \quad \text{Factor out GCF.} \\ x(x+2)(x-2) &= 0 \quad \text{Apply difference of squares.} \end{aligned}$$

Note that we now have a product of three factors that equals zero. The zero product property says that at least one of these factors must equal zero.

$$x = 0$$

or

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

or

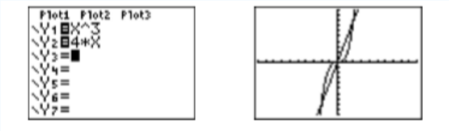
$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Hence, the solutions of $x^3 = 4x$ are $x = 0$, $x = -2$, and $x = 2$.

Graphical solution

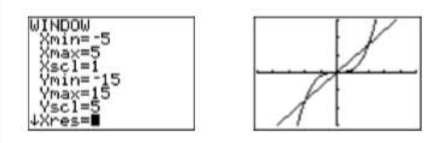
Load $y = x^3$ and $y = 4x$ into $\mathtt{Y1}$ and $\mathtt{Y2}$in the **Y= menu** of your calculator. Select **6:ZStandard** from the **ZOOM** menu to produce the graph in Figure 6.5.1.

Although the image in Figure 6.5.1 shows all three points of intersection, adjusting the **WINDOW** parameters as shown in Figure 6.5.2, then pressing the **GRAPH** button will produce a nicer view of the points of intersection, as shown in the figure on the right in Figures 6.5.2.



```
Plot1  Plot2  Plot3
\Y1=X^3
\Y2=4*X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

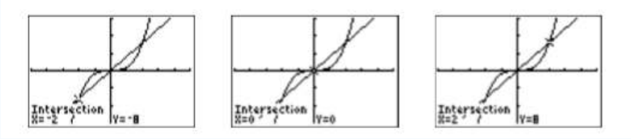
Figure 6.5.1: Sketching $y = x^3$ and $y = 4x$.



```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-15
 Ymax=15
 Yscl=5
↓Xres=1
```

Figure 6.5.2: Adjusting the viewing window.

Use the **5:intersect** tool from the **CALC** menu to find the three points of intersection. Press the **ENTER** key in response to “First curve,” then press **ENTER** again in response to “Second curve,” then use the left-arrow key to move your cursor close to the leftmost point of intersection and press **ENTER** in response to “Guess.” The result is shown in the first image on the left in Figure 6.5.3. Repeat the process to find the remaining points of intersection. The results are shown in the last two images in Figure 6.5.3.



```
Intersection
X=-2        Y=-8
```

```
Intersection
X=0         Y=0
```

```
Intersection
X=2         Y=8
```

Figure 6.5.3: Finding the points of intersection.

Thus, the graphical solutions are $x = -2$, $x = 0$, and $x = 2$.

Reporting the solution on your homework:

Duplicate the image in your calculator’s viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 6.5.4).
- Place your WINDOW parametersat the end of each axis (see Figure 6.5.4).
- Label the graph with its equation (see Figure 6.5.4).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^3 = 4x$ (see Figure 6.5.4).

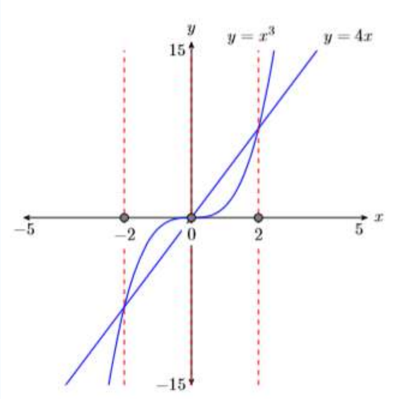


Figure 6.5.4: Reporting your graphical solution on your homework.

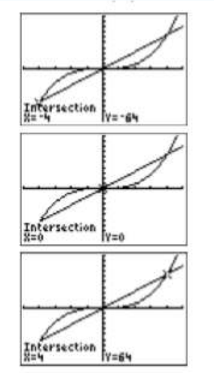
Finally, note that the graphical solutions $x = -2$, $x = 0$, and $x = 2$ match our algebraic solutions exactly.

Exercise 6.5.14

Solve the equation $x^3 = 16x$ both algebraically and graphically, then compare your answers.

Answer

-4, 0, 4



```
Intersection
X=-4          Y=-64
```

```
Intersection
X=0           Y=0
```

```
Intersection
X=4           Y=64
```

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6.6: Factoring Strategy

When you are concentrating on factoring problems of a single type, after doing a few you tend to get into a rhythm, and the remainder of the exercises, because they are similar, seem to flow. However, when you encounter a mixture of factoring problems of different types, progress is harder. The goal of this section is to set up a strategy to follow when attacking a general factoring problem.

If it hasn't already been done, it is helpful to arrange the terms of the given polynomial in some sort of order (descending or ascending). Then you want to apply the following guidelines.

Factoring Strategy

These steps should be followed in the order that they appear.

1. Factor out the greatest common factor (GCF).
2. Look for a special form.
   a. If you have two perfect squares separated by a minus sign, use the difference of squares pattern to factor:

   $$a^2 - b^2 = (a+b)(a-b)$$

   b. If you have a trinomial whose first and last terms are perfect squares, you should suspect that you have a perfect square trinomial. Take the square roots of the first and last terms and factor as follows.

   $$a^2 + 2ab + b^2 = (a+b)^2$$

   Be sure to check that the middle term is correct.
3. If you have a trinomial of the form $ax^2 + bx + c$, use the ac-method to factor.
4. If you have a four-term expression, try to factor by grouping.

Once you've applied the above strategy to the given polynomial, it is quite possible that one of your resulting factors will factor further. Thus, we have the following rule.

Factor completely

The factoring process is not complete until none of your remaining factors can be factored further. This is the meaning of the phrase, "factor completely."

Finally, a very good word of advice.

Check your factoring by multiplying

Once you've factored the given polynomial completely, it is a very good practice to check your result. If you multiply to find the product of your factors, and get the original given polynomial as a result, then you know your factorization is correct.

It's a bit more work to check your factorization, but it's worth the effort. It helps to eliminate errors and also helps to build a better understanding of the factoring process. Remember, factoring is "unmultiplying," so the more you multiply, the better you get at factoring.

Let's see what can happen when you don't check your factorization!

Warning! The following solution is incorrect!

Factor: $2x^4 + 8x^2$

Solution: Factor out the GCF

$$\begin{aligned} 2x^4 + 8x^2 &= 2x^2(x^2+4) \\ &= 2x^2(x+2)^2 \end{aligned}$$

Note that this student did not bother to check his factorization. Let's do that for him now.

Check: Multiply to check. Remember, when squaring a binomial, there is a middle term.

$$\begin{aligned} 2x^2(x+2)^2 &= 2x^2(x^2+4x+4) \\ &= 2x^4+8x^3+8x^2 \end{aligned}$$

This is not the same as the original polynomial $2x^4+8x^2$, so the student's factorization is incorrect. Had the student performed this check, he might have caught his error, provided of course, that he multiplies correctly during the check.

The correct factorization follows.

$$2x^4+8x^2 = 2x^2(x^2+4)$$

The sum of squares does not factor, so we are finished.

Check: Multiply to check.

$$2x^2(x^2+4) = 2x^4+8x^2$$

This is the same as the original polynomial $2x^4+8x^2$, so this factorization is correct.

Example 6.6.1

Factor completely: $-3x^6+3x^2$

Solution

The first rule of factoring is "Factor out the GCF." The GCF of $-3x^6$ and $3x^2$ is $3x^2$, so we could factor out $3x^2$.

$$-3x^6+3x^2 = 3x^2(-x^4+1)$$

This is perfectly valid, but we don't like the fact that the second factor starts with $-x^4$. Let's factor out $-3x^2$ instead.

$$-3x^6+3x^2 = -3x^2(x^4-1)$$

The second factor is the difference of two squares. Take the square roots, separating one pair with a plus sign, one pair with a minus sign.

$$= -3x^2(x^2+1)(x^2-1)$$

The sum of squares does not factor. But the last factor is the difference of two squares. Take the square roots, separating one pair with a plus sign, one pair with a minus sign.

$$= -3x^2(x^2+1)(x+1)(x-1)$$

Check: Multiply to check the result.

$$\begin{aligned} -3x^2(x^2+1)(x+1)(x-1) &= -3x^2(x^2+1)(x^2-1) \\ &= -3x^2(x^4-1) \\ &= -3x^6+3x^2 \end{aligned}$$

The factorization checks.

Exercise 6.6.1

Factor completely: $-4x^7+64x^3$

Answer

$-4x^3(x^2+4)(x+2)(x-2)$

Example 6.6.2

Factor completely: $x^3y+9xy^3+6x^2y^2$

Solution

The first rule of factoring is “Factor out the GCF.” The GCF of x^3y, $9xy^3$, and $6x^2y^2$ is xy, so we factor out xy.

$$x^3y+9xy^3+6x^2y^2=xy(x^2+9y^2+6xy)$$

Let’s order that second factor in descending powers of x.

$$=xy(x^2+6xy+9y^2)$$

The first and last terms of the trinomial factor are perfect squares. We suspect we have a perfect square trinomial, so we take the square roots of the first and last terms, check the middle term, and write:

$$=xy(x+3y)^2$$

Thus, $x^3y+9xy^3+6x^2y^2=xy(x+3y)^2$.

Check: Multiply to check the result.

$$\begin{aligned}xy(x+3y)^2&=xy(x^2+6xy+9y^2)\\&=x^3y+6x^2y^2+9xy^3\end{aligned}$$

Except for the order, this result is the same as the given polynomial. The factorization checks.

Exercise 6.6.2

Factor completely: $3a^2b^4+12a^4b^2-12a^3b^3$

Answer

$3a^2b^2(2a-b)^2$

Example 6.6.3

Factor completely: $2x^3-48x+20x^2$

Solution

In the last example, we recognized a need to rearrange our terms after we pulled out the GCF. This time, let’s arrange our terms in descending powers of x right away.

$$2x^3-48x+20x^2=2x^3+20x^2-48x$$

Now, let’s factor out the GCF.

$$=2x(x^2+10x-24)$$

The last term of the trinomial factor is not a perfect square. Let’s move to the ac-method to factor. The integer pair $-2, 12$ has a product equal to $ac=-24$ and a sum equal to $b=10$. Because the coefficient of x^2 is one, this is a “drop in place” situation. We drop our pair in place and write:

$$=2x(x-2)(x+12)$$

Thus, $2x^3-48x+20x^2=2x(x-2)(x+12)$.

Check: Multiply to check the result. We use the FOIL method shortcut and mental calculations to speed things up.

$$\begin{aligned}2x(x-2)(x+12)&=2x(x^2+10x-24)\\&=2x^3+20x^2-48x\end{aligned}$$

Except for the order, this result is the same as the given polynomial. The factorization checks.

Exercise 6.6.3

Add exercises text here.

Answer

$-3x^2(x-4)(x-5)$

Example 6.6.4

Factor completely: $2a^2-13ab-24b^2$

Solution

There is no common factor we can factor out. We have a trinomial, but the first and last terms are not perfect squares, so let's apply the ac-method. Ignoring the variables for a moment, we need an integer pair whose product is $ac=-48$ and whose sum is -13. The integer pair $3,-16$ comes to mind (if nothing comes to mind, start listing integer pairs). Break up the middle term into a sum of like terms using the integer pair $3,-16$, then factor by grouping

$$\begin{aligned}2a^2-13ab-24b^2&=2a^2+3ab-16ab-24b^2\\&=a(2a+3b)-8b(2a+3b)\\&=(a-8b)(2a+3b)\end{aligned}$$

Thus, $2a^2-13ab-24b^2=(a-8b)(2a+3b)$.

Check: Multiply to check the result. We use the FOIL method shortcut and mental calculations to speed things up.

$$(a-8b)(2a+3b)=2a^2-13ab-24b^2$$

This result is the same as the given polynomial. The factorization checks.

Exercise 6.6.4

Factor completely: $8x^2+14xy-15y^2$

Answer

$(2x+5y)(4x-3y)$

Example 6.6.5

Factor completely: $30x^4+38x^3-20x^2$

Solution

The first step is to factor out the GCF, which in this case is $2x^2$.

$$30x^4+38x^3-20x^2=2x^2(15x^2+19x-10)$$

The first and last terms of the trinomial factor are not perfect squares, so let's move again to the ac-method. Comparing $15x^2+19x-10$ with ax^2+bx+c, note that $ac=(15)(-10)=-150$. We need an integer pair whose product is -150 and whose sum is 19. The integer pair -6 and 25 satisfies these requirements. Because $a\neq 1$, this is not a "drop in place" situation, so we need to break up the middle term as a sum of like terms using the pair -6 and 25.

$$=2x^2(15x^2-6x+25x-10)$$

Factor by grouping. Factor $3x$ out of the first two terms and 5 out of the third and fourth terms.

$$=2x^2(3x(5x-2)+5(5x-2))$$

Finally, factor out the common factor $5x-2$.

$$=2x^2(3x+5)(5x-2)$$

Thus, $30x^4+38x^3-20x^2=2x^2(3x+5)(5x-2)$.

Check: Multiply to check the result. Use the FOIL method to first multiply the binomials.

$$2x^2(3x+5)(5x-2) = 2x^2(15x^2+19x-10)$$

Distribute the $2x^2$.

$$= 30x^4+38x^3-20x^2$$

This result is the same as the given polynomial. The factorization checks.

Exercise 6.6.5

Factor completely: $36x^3+60x^2+9x$

Answer

$3x(6x+1)(2x+3)$

Example 6.6.6

Factor completely: $8x^5+10x^4-72x^3-90x^2$

Solution

Each of the terms is divisible by $3x^3$. Factor out $3x^3$.

$$15x^6-33x^5-240x^4+528x^3 = 3x^3[5x^3-11x^2-80x+176]$$

The second factor is a four-term expression. Factor by grouping.

$$\begin{aligned} &= 3x^3[x^2(5x-11)-16(5x-11)] \\ &= 3x^3(x^2-16)(5x-11) \end{aligned}$$

The factor x^2-16 is a difference of two squares. Take the square roots, separate one pair with a plus, one pair with a minus.

$$= 3x^3(x+4)(x-4)(5x-11)$$

Thus, $15x^6-33x^5-240x^4+528x^3 = 3x^3(x+4)(x-4)(5x-11)$.

Check: Multiply to check the result.

$$\begin{aligned} 3x^3(x+4)(x-4)(5x-11) &= 3x^3(x^2-16)(5x-11) \\ &= 3x^3(5x^3-11x^2-80x+176 \\ &= 15x^6-33x^5-240x^4+528x^3 \end{aligned}$$

This result is the same as the given polynomial. The factorization checks.

Exercise 6.6.6

Factor completely: $15x^6-33x^5-240x^4+528x^3$

Answer

$2x^2(x-3)(x+3)(4x+5)$

Using the Calculator to Assist the ac-Method

When using the ac-method to factor ax^2+bx+c and ac is a very large number, then it can be difficult to find a pair whose product is ac and whose sum in b. For example, consider the trinomial:

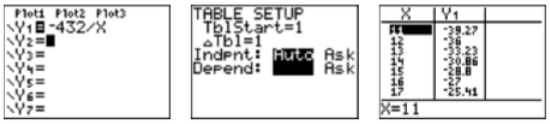
$$12y^2-11y-36$$

We need an integer pair whose product is $ac=-432$ and whose sum is $b=-11$. We begin listing integer pair possibilities, but the process quickly becomes daunting.

$$\begin{gathered}1,-432\\2,-216\\\ldots\end{gathered}$$

Note that the numbers in the second column are found by dividing $ac=-432$ by the number in the first column. We'll now use this fact and the **TABLE** feature on our calculator to pursue the desired integer pair.

1. Enter the expression $-432/X$ into $\mathbb{Y}1$ in the **Y= menu** (see the first image in Figure $6.6.1$).
2. Above the **WINDOW** button you'll see **TBLSET**. Use the **2nd** key, then press the **WINDOW** button to access the menu shown in the second image of Figure $6.6.1$. Set **TblStart=1**, $\triangle\text{Tbl}=1$, then highlight **AUTO** for both the independent and dependent variables.
3. Above the **GRAPH** button you'll see **TABLE**. Use the **2nd** key, then press the **GRAPH** button to access the table shown in the third image in Figure $6.6.1$. Use the up- and down-arrow keys to scroll through the contents of the table. Note that you can ignore most of the pairs, because they are not both integers. Pay attention only when they are both integers. In this case, remember that you are searching for a pair whose sum is $b=-11$. Note that the pair $16,-27$ shown in the third image of Figure $6.6.1$ is the pair we seek.



```
Plot1 Plot2 Plot3
\Y1■-432/X
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
TABLE SETUP
 TblStart=1
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

```
  X   | Y1
11    | -39.27
12    | -36
13    | -33.23
14    | -30.86
15    | -28.8
16    | -27
17    | -25.41
X=11
```

Figure $6.6.1$: Using the **TABLE** feature to assist the ac-method.

Now we can break the middle term of $12y^2-11y-36$ into a sum of like terms using the ordered pair $16,-27$, then factor by grouping.

$$\begin{aligned}12y^2-11y-36&=12y^2+16y-27y-36\\&=4y(3y+4)-9(3y+4)\\&=(4y-9)(3y+4)\end{aligned}$$

Check: Use the FOIL method shortcut and mental calculations to multiply.

$$(4y-9)(3y+4)=12y^2-11y-36$$

The factorization checks.

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6.7: Applications of Factoring

In this section we will solve applications whose solutions involve factoring. Let's begin.

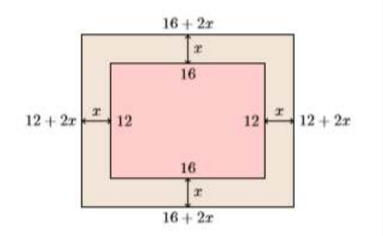
Example 6.7.1

A rectangular canvas picture measures 12 inches by 16 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 396 square inches. Find the uniform width of the frame.

Solution

We follow the *Requirements for Word Problem Solutions*.

1. **Set up a variable dictionary:** A carefully labeled figure will help us maintain our focus. We'll let x represent the uniform width of the frame.



2. **Set up an equation:** If the inner rectangular dimensions are 16 inches by 12 inches, adding a frame of uniform width x makes the dimensions of the frame plus canvas $16+2x$ inches by $12+2x$ inches. The total area is found by multiplying these outer dimensions, $A=(16+2x)(12+2x)$. If the total area is $A=396$ square inches, then we have the following equation.

$$(16+2x)(12+2x)=396$$

3. **Solve the equation:** We start by expanding the right-hand side of the equation.

$$\begin{aligned}(16+2x)(12+2x)&=396\\ 192+56x+4x^2&=396\end{aligned}$$

The resulting equation is nonlinear. Make one side zero.

$$4x^2+56x-204=0$$

We could factor out 4 on the left-hand side, but since there is a zero on the right-hand side of the equation, it's a bit easier to simply divide both sides by 4. Note how we divide each term on the left-hand side by the number 4.

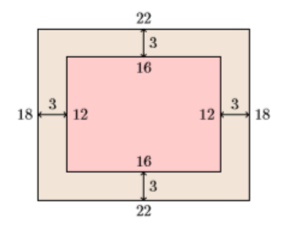
$$x^2+14x-51=0$$

We need an integer pair whose product is $ac=-51$ and whose sum is $b=14$. The integer pair $-3, 17$ comes to mind. Since the coefficient of x^2 is one, we can simply "drop in place" our ordered pair.

$$(x-3)(x+17)=0$$

Thus, the solutions are $x=3$ and $x=-17$.

4. **Answer the question:** The uniform width of the frame cannot be -17 inches, so we eliminate this solution from consideration. Hence, the uniform width of the frame is 3 inches.
5. **Look back:** If the uniform width of the frame is 3 inches, then the final dimensions will look like the following.



Thus, the combined area of the frame plus canvas is $A = (18)(22)$, or $A = 396$ square inches, the area given in the problem statement. Our solution is correct.

Exercise 6.7.1

A rectangular canvas picture measures 7 inches by 11 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 117 square inches. Find the uniform width of the frame.

Answer

1 inch

Example 6.7.2

A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 120t \tag{6.7.1}$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 288t + 640 \tag{6.7.2}$$

where t is the amount of time (in seconds) that has passed since the projectile's release.

a. How much time passes before the projectile splashes into the ocean?
b. At that time, how far is the projectile from the base of the cliff?

Solution

We follow the *Requirements for Word Problem Solutions*.

1. **Set up a variable dictionary:** The variables are already set up for us.

$$\begin{aligned} x &= \text{Distance from base of the cliff (in feet).} \\ y &= \text{Height above sea level (in feet).} \\ t &= \text{Time since projectile's release (in seconds).} \end{aligned}$$

2. **Set up an equation:** The equations are already set up (see Equation 6.7.1 and Equation 6.7.2).
3. **Solve the equation:** When the projectile splashes into the ocean, its height above sea level at that moment is $y = 0$ feet. Substitute 0 for y in Equation 6.7.2 and solve the resulting equation for t.

$$\begin{aligned} y &= -16t^2 + 288t + 640 \\ 0 &= -16t^2 + 288t + 640 \end{aligned}$$

We could factor out -16, but since the left-hand side of this equation is zero, it's a bit easier to divide both sides by -16. Note how we divide each term on the right-hand side by -16.

$$0 = t^2 - 18t - 40$$

We need a pair of integers so that their product is $ac = -40$ and their sum is -18. The integer pair $2, -20$ comes to mind. Since the coefficient of t^2 is one, we can "drop in place" our integer pair.

$$0 = (t + 2)(t - 20)$$

Hence, the solutions are $t=-2$ and $t=20$.

4. **Answer the question:** In answering question (a), the solution $t=-2$ seconds makes no sense. Thus, the projectile splashes into the ocean at $t=20$ seconds. In addressing question (b), to find the projectile's distance from the base of the cliff at this moment, substitute $t=20$ in Equation 6.7.1.

$$\begin{aligned}x&=120t\\x&=120(20)\\x&=2400\end{aligned}$$

Hence, at the moment the projectile splashes into the ocean, it is $2,400$ feet from the base of the cliff.

5. **Look back:** The best we can do here is check our solution $t=20$ in Equation 6.7.2.

$$\begin{aligned}y&=-16t^2+288t+640\\y&=-16(20)2+288(20)+640\\y&=-6400+5760+640\\y&=0\end{aligned}$$

Indeed, at $t=20$, the projectile does splash into the ocean.

Exercise 6.7.2

A projectile's height (in feet) is given by the equation $y=-16t^2+144t+576$, where time t is measured in seconds. How much time passes before the projectile hits the ground?

Answer

12 seconds

Example 6.7.3

The product of two consecutive even integers is 728. Find the integers.

Solution

We follow the Requirements for Word Problem Solutions.

1. **Set up a variable dictionary:** Let k represent the first even integer. Then $k+2$ represents the next consecutive even integer.
2. **Set up an equation:** The product of the integers is 728. Hence, we have the following equation.

$$k(k+2)=728$$

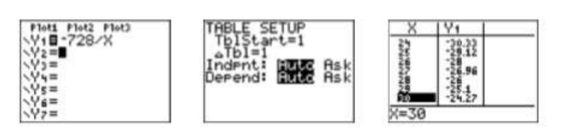
3. **Solve the equation:** Expand the left-hand side of the equation.

$$k^2+2k=728$$

The equation is nonlinear. Make one side zero.

$$k^2+2k-728=0$$

See Using the Calculator to Assist the ac-Method. We need an integer pair whose product is $ac=-728$ and whose sum is $b=2$. Enter $-728/X$ in Y1, then set up the table (see Figure 6.7.1).



```
Plot1 Plot2 Plot3
\Y1=-728/X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
TABLE SETUP
 TblStart=1
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

```
  X   | Y1
 24   | -30.33
 25   | -29.12
 26   | -28
 27   | -26.96
 28   | -26
 29   | -25.1
 30   | -24.27
X=30
```

Figure 6.7.1: Load ac/X, or $-728/X$ into Y1 in the **Y= menu**.

Use the up- and down-arrow keys to scroll. Note that $28,-26$ is the desired pair. Because the coefficient of k^2 is one, we can "drop in place" the ordered pair.

$$0=(k+28)(k-26)$$

Hence, the solutions are $k=-28$ and $k=26$.

4. **Answer the question:** If $k=-28$, the next consecutive even integer is $k+2=-26$. Secondly, if $k=26$, the next consecutive even integer is $k+2=28$.
5. **Look back:** Our first pair is -28 and -26. They have the required product of 728. Our second pair is 26 and 28. Their product is also 728. Both solutions check!

Exercise 6.7.3

The product of two consecutive positive odd integers is 483. Find the integers.

Answer

21 and 23

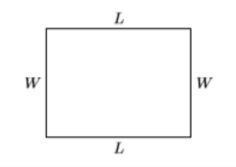
Example 6.7.4

A rectangle has perimeter 54 feet and area 180 square feet. Find the dimensions of the rectangle.

Solution

We follow the Requirements for Word Problem Solutions.

1. **Set up a variable dictionary:** A sketch will help us keep our focus. Let L represent the length of the rectangle and let W represent its width.



2. **Set up an equation:** The perimeter is 54 feet, so $2W+2L=54$, or dividing both sides by 2:

$$W+L=27 \tag{6.7.3}$$

The area is 180 square feet, so:

$$LW=180 \tag{6.7.4}$$

3. **Solve the equation:** The system of equations (equations 6.7.3 and 6.7.4) can be solved using the substitution method. First, solve Equation 6.7.3 for W:

$$W=27-L \tag{6.7.5}$$

Substitute Equation 6.7.5 in Equation 6.7.4, expand, then make one side zero.

$$\begin{aligned} L(27-L) &= 180 \\ 27L-L^2 &= 180 \\ 0 &= L^2-27L+180 \end{aligned}$$

The integer pair $-12, -15$ has product $ac=180$ and sum $b=-27$. Moreover, the coefficient of L^2 is one, so we can “drop in place” our integer pair.

$$0=(L-12)(L-15)$$

Hence, the solutions are $L=12$ and $L=15$.

4. **Answer the question:** Two possibilities for the width.
   i. Substitute $L=12$ in 6.7.5.

$$\begin{aligned} W &= 27-L \\ W &= 27-12 \\ W &= 15 \end{aligned}$$

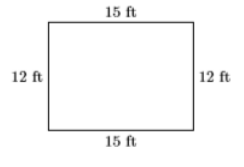
   ii. Substitute $L=15$ in 6.7.5.

$$\begin{aligned}W &= 27 - L\\W &= 27 - 15\\W &= 12\end{aligned}$$

Both answers give the same 15 by 12 foot rectangle, but we usually think of the term “length” as the longer side of the rectangle.

So let’s go with the length is $L = 15$ feet and the width is $W = 12$ feet.

5. **Look back:** Let’s add $L = 15$ feet and $W = 12$ feet to a diagram.



If we add the dimensions around the rectangle, the perimeter is $P = 15 + 12 + 15 + 12$, or $P = 54$ feet, the perimeter required in the problem statement.

Next, if we multiply the dimensions, then $A = (15)(12)$, or $A = 180$ square feet, the area required in the problem statement. Our solution is correct!

Exercise $6.7.4$

A rectangle has perimeter 62 feet and area 234 square feet. Find the dimensions of the rectangle.

Answer

length = 18 feet and width = 13 feet

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6.E: Factoring (Exercises)

6.1: The Greatest Common Factor

In Exercises 1-6, list all positive divisors of the given number, in order, from smallest to largest.

1) 42

Answer

$\{1,2,3,6,7,14,21,42\}$

2) 60

3) 44

Answer

$\{1,2,4,11,22,44\}$

4) 85

5) 51

Answer

$\{1,3,17,51\}$

6) 63

In Exercises 7-12, list all common positive divisors of the given numbers, in order, from smallest to largest.

7) 36 and 42

Answer

$\{1,2,3,6\}$

8) 54 and 30

9) 78 and 54

Answer

$\{1,2,3,6\}$

10) 96 and 78

11) 8 and 76

Answer

$\{1,2,4\}$

12) 99 and 27

In Exercises 13-18, state the greatest common divisor of the given numbers.

13) 76 and 8

Answer

4

14) 84 and 60

15) 32 and 36

Answer

4

16) 64 and 76

17) 24 and 28

Answer

4

18) 63 and 27

In Exercises 19-24, use prime factorization to help calculate the greatest common divisor of the given numbers.

19) 600 and 1080

Answer

120

20) 150 and 120

21) 1800 and 2250

Answer

450

22) 540 and 150

23) 600 and 450

Answer

150

24) 4500 and 1800

In Exercises 25-36, find the greatest common factor of the given expressions.

25) $16b^4$ and $56b^9$

Answer

$8b^4$

26) $28s^2$ and $36s^4$

27) $35z^2$ and $49z^7$

Answer

$7z^2$

28) $24w^3$ and $30w^8$

29) $56x^3y^4$ and $16x^2y^5$

Answer

$8x^2y^4$

30) $35b^5c^3$ and $63b^4c^4$

31) $24s^4t^5$ and $16s^3t^6$

Answer

$8s^3t^5$

32) $10v^4w^3$ and $8v^3w^4$

33) $18y^7$, $45y^6$, and $27y^5$

Answer

$9y^5$

34) $8r^7$, $24r^6$, and $12r^5$

35) $9a^6$, $6a^5$, and $15a^4$

Answer

$3a^4$

36) $15a^5$, $24a^4$, and $24a^3$

In Exercises 37-52, factor out the GCF from each of the given expressions.

37) $25a^2+10a+20$

Answer

$5\left(5a^2+2a+4\right)$

38) $40c^2+15c+40$

39) $35s^2+25s+45$

Answer

$5\left(7s^2+5s+9\right)$

40) $45b^2+20b+35$

41) $16c^3+32c^2+36c$

Answer

$4c\left(4c^2+8c+9\right)$

42) $12b^3+12b^2+18b$

43) $42s^3+24s^2+18s$

Answer

$6s\left(7s^2+4s+3\right)$

44) $36y^3+81y^2+36y$

45) $35s^7+49s^6+63s^5$

Answer

$7s^5\left(5s^2+7s+9\right)$

46) $35s^7+56s^6+56s^5$

47) $14b^7+35b^6+56b^5$

Answer

$7b^5\left(2b^2+5b+8\right)$

48) $45x^5 + 81x^4 + 45x^3$

49) $54y^5z^3 + 30y^4z^4 + 36y^3z^5$

Answer

$6y^3z^3\left(9y^2 + 5yz + 6z^2\right)$

50) $42x^4y^2 + 42x^3y^3 + 54x^2y^4$

51) $45s^4t^3 + 40s^3t^4 + 15s^2t^5$

Answer

$5.8^2t^3\left(9s^2 + 8st + 3t^2\right)$

52) $20v^6w^3 + 36v^5w^4 + 28v^4w^5$

In Exercises 53-60, factor out the GCF from each of the given expressions.

53) $7w(2w-3) - 8(2w-3)$

Answer

$(7w-8)(2w-3)$

54) $5s(8s-1) + 4(8s-1)$

55) $9r(5r-1) + 8(5r-1)$

Answer

$(9r+8)(5r-1)$

56) $5c(4c-7) + 2(4c-7)$

57) $48a(2a+5) - 42(2a+5)$

Answer

$6(2a+5)(8a-7)$

58) $40v(7v-4) + 72(7v-4)$

59) $56a(2a-1) - 21(2a-1)$

Answer

$7(2a-1)(8a-3)$

60) $48r(5r+3) - 40(5r+3)$

In Exercises 61-68, factor by grouping. Do not simplify the expression before factoring.

61) $x^2 + 2x - 9x - 18$

Answer

$(x-9)(x+2)$

62) $x^2 + 6x - 9x - 54$

63) $x^2 + 3x + 6x + 18$

Answer

$(x+6)(x+3)$

64) $x^2+8x+7x+56$

65) $x^2-6x-3x+18$

Answer

$(x-3)(x-6)$

66) $x^2-3x-9x+27$

67) $x^2-9x+3x-27$

Answer

$(x+3)(x-9)$

68) $x^2-2x+7x-14$

In Exercises 69-76, factor by grouping. Do not simplify the expression before factoring.

69) $8x^2+3x-56x-21$

Answer

$(x-7)(8x+3)$

70) $4x^2+9x-32x-72$

71) $9x^2+36x-5x-20$

Answer

$(9x-5)(x+4)$

72) $7x^2+14x-8x-16$

73) $6x^2-7x-48x+56$

Answer

$(x-8)(6x-7)$

74) $8x^2-7x-72x+63$

75) $2x^2+12x+7x+42$

Answer

$(2x+7)(x+6)$

76) $7x^2+28x+9x+36$

6.2: Solving Nonlinear Equations

In Exercises 1-8, solve the given equation for x.

1) $(9x+2)(8x+3)=0$

Answer

$$x=-\frac{2}{9},-\frac{3}{8}$$

2) $(2x-5)(7x-4)=0$

3) $x(4x+7)(9x-8)=0$

Answer

$x=0,-\frac{7}{4},\frac{8}{9}$

4) $x(9x-8)(3x+1)=0$

5) $-9x(9x+4)=0$

Answer

$x=0,-\frac{4}{9}$

6) $4x(3x-6)=0$

7) $(x+1)(x+6)=0$

Answer

$x=-1,-6$

8) $(x-4)(x-1)=0$

In Exercises 9-18, given that you are solving for x, state whether the given equation is linear or nonlinear. Do not solve the equation.

9) $x^2+7x=9x+63$

Answer

Nonlinear

10) $x^2+9x=4x+36$

11) $6x-2=5x-8$

Answer

Linear

12) $-5x+5=-6x-7$

13) $7x^2=-2x$

Answer

Nonlinear

14) $4x^2=-7x$

15) $3x^2+8x=-9$

Answer

Nonlinear

16) $5x^2-2x=-9$

17) $-3x+6=-9$

Answer

Linear

18) $8x-5=3$

In Exercises 19-34, solve each of the given equations for x.

19) $3x+8=9$

Answer

$$\frac{1}{3}$$

20) $3x+4=2$

21) $9x^2=-x$

Answer

$$x=0,-\frac{1}{9}$$

22) $6x^2=7x$

23) $3x+9=8x+7$

Answer

$$\frac{2}{5}$$

24) $8x+5=6x+4$

25) $8x^2=-2x$

Answer

$$x=0,-\frac{1}{4}$$

26) $8x^2=18x$

27) $9x+2=7$

Answer

$$\frac{5}{9}$$

28) $3x+2=6$

29) $9x^2=6x$

Answer

$$x=0,\frac{2}{3}$$

30) $6x^2=-14x$

31) $7x^2=-4x$

Answer

$$x=0,-\frac{4}{7}$$

32) $7x^2=-9x$

33) $7x+2=4x+7$

Answer

$$\frac{5}{3}$$

34) $4x+3=2x+8$

In Exercises 35-50, factor by grouping to solve each of the given equations for x.

35) $63x^2+56x+54x+48=0$

Answer

$x=-\frac{6}{7},-\frac{8}{9}$

36) $27x^2+36x+6x+8=0$

37) $16x^2-18x+40x-45=0$

Answer

$x=-\frac{5}{2},\frac{9}{8}$

38) $42x^2-35x+54x-45=0$

39) $45x^2+18x+20x+8=0$

Answer

$x=-\frac{4}{9},-\frac{2}{5}$

40) $18x^2+21x+30x+35=0$

41) $x^2+10x+4x+40=0$

Answer

$x=-4,-10$

42) $x^2+11x+10x+110=0$

43) $x^2+6x-11x-66=0$

Answer

$x=11,-6$

44) $x^2+6x-2x-12=0$

45) $15x^2-24x+35x-56=0$

Answer

$x=-\frac{7}{3},\frac{8}{5}$

46) $12x^2-10x+54x-45=0$

47) $x^2+2x+9x+18=0$

Answer

$x=-9,-2$

48) $x^2+8x+4x+32=0$

49) $x^2+4x-8x-32=0$

Answer

$x=8,-4$

50) $x^2+8x-5x-40=0$

In Exercises 51-54, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 5:intersect utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.2.7.

51) $x^2 = -4x$

Answer

$x = -4, 0$

52) $x^2 = 6x$

53) $x^2 = 5x$

Answer

$x = 0, 5$

54) $x^2 = -6x$

In Exercises 55-58, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 2:zero utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.2.8.

55) $x^2 + 7x = 0$

Answer

$x = -7, 0$

56) $x^2 - 8x = 0$

57) $x^2 - 3x = 0$

Answer

$x = 0, 3$

58) $x^2 + 2x = 0$

6.3: Factoring ax² + bx + c when a =1

In Exercises 1-6, compare the given trinomial with $ax^2 + bx + c$, then list ALL integer pairs whose product equals ac. Circle the pair whose sum equals b, then use this pair to help factor the given trinomial.

1) $x^2 + 7x - 18$

Answer

$(x - 2)(x + 9)$

2) $x^2 + 18x + 80$

3) $x^2 - 10x + 9$

Answer

$(x - 1)(x - 9)$

4) $x^2 + 12x + 27$

5) $x^2 + 14x + 45$

Answer

$(x + 5)(x + 9)$

6) $x^2+9x+20$

In Exercises 7-12, compare the given trinomial with ax^2+bx+c, then begin listing integer pairs whose product equals ac. Cease the list process when you discover a pair whose sum equals b, then circle and use this pair to help factor the given trinomial.

7) $x^2-16x+39$

Answer

$(x-3)(x-13)$

8) $x^2-16x+48$

9) $x^2-26x+69$

Answer

$(x-3)(x-23)$

10) $x^2-22x+57$

11) $x^2-25x+84$

Answer

$(x-4)(x-21)$

12) $x^2+13x-30$

In Exercises 13-18, compare the given trinomial with ax^2+bx+c, then compute ac. Try to mentally discover the integer pair whose product is ac and whose sum is b. Factor the trinomial by "dropping this pair in place."

Note: If you find you cannot identify the pair mentally, begin listing integer pairs whose product equals ac, then cease the listing process when you encounter the pair whose sum equals b.

13) $x^2-13x+36$

Answer

$(x-4)(x-9)$

14) x^2+x-12

15) $x^2+10x+21$

Answer

$(x+3)(x+7)$

16) $x^2-17x+66$

17) x^2-4x-5

Answer

$(x+1)(x-5)$

18) $x^2-20x+99$

In Exercises 19-24, use an algebraic technique to solve the given equation.

19) $x^2=-7x+30$

Answer

$x=3,-10$

20) $x^2 = -2x + 35$

21) $x^2 = -11x - 10$

Answer

$x = -1, -10$

22) $x^2 = x + 72$

23) $x^2 = -15x - 50$

Answer

$x = -5, -10$

24) $x^2 = -7x - 6$

In Exercises 25-30, use an algebraic technique to solve the given equation.

25) $60 = x^2 + 11x$

Answer

$x = 4, -15$

26) $-92 = x^2 - 27x$

27) $-11 = x^2 - 12x$

Answer

$x = 1, 11$

28) $80 = x^2 - 16x$

29) $56 = x^2 + 10x$

Answer

$x = 4, -14$

30) $66 = x^2 + 19x$

In Exercises 31-36, use an algebraic technique to solve the given equation.

31) $x^2 + 20 = -12x$

Answer

$x = -2, -10$

32) $x^2 - 12 = 11x$

33) $x^2 - 36 = 9x$

Answer

$x = -3, 12$

34) $x^2 + 6 = 5x$

35) $x^2 + 8 = -6x$

Answer

$x = -2, -4$

36) $x^2+77=18x$

In Exercises 37-40, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 5:intersect utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.3.6.

37) $x^2=x+12$

Answer

$x=-3,4$

38) $x^2=20-x$

39) $x^2+12=8x$

Answer

$x=2,6$

40) $x^2+7=8x$

In Exercises 41-44, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 2:zero utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.3.7.

41) $x^2-6x-16=0$

Answer

$x=8,-2$

42) $x^2+7x-18=0$

43) $x^2+10x-24=0$

Answer

$x=-12,2$

44) $x^2-9x-36=0$

6.4: Factoring ax² + bx + c when a≠1

In Exercises 1-6, compare the given trinomial with ax^2+bx+c, then list ALL integer pairs whose product equals ac. Circle the pair whose sum equals b, then use this pair to help factor the given trinomial.

1) $6x^2+13x-5$

Answer

$(2x+5)(3x-1)$

2) $3x^2-19x+20$

3) $4x^2-x-3$

Answer

$(x-1)(4x+3)$

4) $6x^2-23x+7$

5) $3x^2+19x+28$

Answer

$(x+4)(3x+7)$

6) $2x^2-9x-18$

In Exercises 7-12, compare the given trinomial with ax^2+bx+c, then begin listing integer pairs whose product equals ac. Cease the list process when you discover a pair whose sum equals b, then circle and use this pair to help factor the given trinomial.

7) $12x^2-23x+5$

Answer

$(3x-5)(4x-1)$

8) $8x^2+22x+9$

9) $6x^2+17x+7$

Answer

$(2x+1)(3x+7)$

10) $4x^2+19x+21$

11) $3x^2+4x-32$

Answer

$(x+4)(3x-8)$

12) $4x^2+x-14$

In Exercises 13-18, compare the given trinomial with ax^2+bx+c, then compute ac. Try to mentally discover the integer pair whose product is ac and whose sum is b. Use this pair to help factor the given trinomial.

Note: If you find you cannot identify the pair mentally, begin listing integer pairs whose product equals ac, then cease the listing process when you encounter the pair whose sum equals b.

13) $3x^2+28x+9$

Answer

$(3x+1)(x+9)$

14) $6x^2+x-1$

15) $4x^2-21x+5$

Answer

$(x-5)(4x-1)$

16) $4x^2-x-14$

17) $6x^2-11x-7$

Answer

$(3x-7)(2x+1)$

18) $2x^2-17x+21$

In Exercises 19-26, factor the trinomial.

19) $16x^5 - 36x^4 + 14x^3$

Answer

$2x^3(2x-1)(4x-7)$

20) $12x^4 - 20x^3 + 8x^2$

21) $36x^4 - 75x^3 + 21x^2$

Answer

$3x^2(3x-1)(4x-7)$

22) $6x^4 - 10x^3 - 24x^2$

23) $6x^4 - 33x^3 + 42x^2$

Answer

$3x^2(x-2)(2x-7)$

24) $15x^3 - 10x^2 - 105x$

25) $16x^4 - 36x^3 - 36x^2$

Answer

$4x^2(x-3)(4x+3)$

26) $40x^4 - 10x^3 - 5x^2$

In Exercises 27-38, use an algebraic technique to solve the given equation.

27) $(10-8)^2 - (7-5)^3$

Answer

$x = 2, -\frac{9}{4}$

28) $2x^2 = 7x - 3$

29) $3x^2 + 16 = -14x$

Answer

$x = -2, -\frac{8}{3}$

30) $2x^2 - 20 = -3x$

31) $3x^2 + 30 = 23x$

Answer

$x = 6, \frac{5}{3}$

32) $6x^2 - 7 = -11x$

33) $-7x - 3 = -6x^2$

Answer

$x = -\frac{1}{3}, \frac{3}{2}$

34) $13x - 45 = -2x^2$

35) $26x-9=-3x^2$

Answer

$$x=-9,\frac{1}{3}$$

36) $-23x+7=-6x^2$

37) $6x^2=-25x+9$

Answer

$$x=\frac{1}{3},-\frac{9}{2}$$

38) $2x^2=13x+45$

In Exercises 39-42, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 2:zero utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.4.5.

39) $2x^2-9x-5=0$

Answer

$$x=-\frac{1}{2},5$$

40) $2x^2+x-28=0$

41) $4x^2-17x-15=0$

Answer

$$x=-\frac{3}{4},5$$

42) $3x^2+14x-24=0$

In Exercises 43-46, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 2:zero utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.4.6.

43) $2x^3=3x^2+20x$

Answer

$$x=0,-\frac{5}{2},4$$

44) $2x^3=3x^2+35x$

45) $10x^3+34x^2=24x$

Answer

$$x=0,-4,\frac{3}{5}$$

46) $6x^3+3x^2=63x$

6.5: Factoring Special Forms

In Exercises 1-8, expand each of the given expressions.

1) $(8r-3t)^2$

Answer

$64r^2-48rt+9t^2$

2) $(6a+c)^2$

3) $(4a+7b)^2$

Answer

$16a^2+56ab+49b^2$

4) $(4s+t)^2$

5) $\left(s^3-9\right)^2$

Answer

s^6-18s^3+81

6) $\left(w^3+7\right)^2$

7) $\left(s^2+6t^2\right)^2$

Answer

$s^4+12s^2t^2+36t^4$

8) $\left(7u^2-2w^2\right)^2$

In Exercises 9-28, factor each of the given expressions.

9) $25s^2+60st+36t^2$

Answer

$(5s+6t)^2$

10) $9u^2+24uv+16v^2$

11) $36v^2-60vw+25w^2$

Answer

$(6v-5w)^2$

12) $49b^2-42bc+9c^2$

13) $a^4+18a^2b^2+81b^4$

Answer

$\left(a^2+9b^2\right)^2$

14) $64u^4-144u^2w^2+81w^4$

15) $49s^4-28s^2t^2+4t^4$

Answer

$\left(7s^2-2t^2\right)^2$

16) $4a^4-12a^2c^2+9c^4$

17) $49b^6-112b^3+64$

Answer

$(7b^3-8)^2$

18) $25x^6-10x^3+1$

19) $49r^6+112r^3+64$

Answer

$(7r^3+8)^2$

20) a^6-16a^3+64

21) $5s^3t-20s^2t^2+20st^3$

Answer

$5st(s-2t)^2$

22) $12r^3t-12r^2t^2+3rt^3$

23) $8a^3c+8a^2c^2+2ac^3$

Answer

$2ac(2a+c)^2$

24) $18x^3z-60x^2z^2+50xz^3$

25) $-48b^3+120b^2-75b$

Answer

$-3b(4b-5)^2$

26) $-45c^3+120c^2-80c$

27) $-5u^5-30u^4-45u^3$

Answer

$-5u^3(u+3)^2$

28) $-12z^5-36z^4-27z^3$

In Exercises 29-36, expand each of the given expressions.

29) $(21c+16)(21c-16)$

Answer

$441c^2-256$

30) $(19t+7)(19t-7)$

31) $(5x+19z)(5x-19z)$

Answer

$25x^2-361z^2$

32) $(11u+5w)(11u-5w)$

33) $\left(3y^{4}+23z^{4}\right)\left(3y^{4}-23z^{4}\right)$

Answer

$9y^{8}-529z^{8}$

34) $\left(5x^{3}+z^{3}\right)\left(5x^{3}-z^{3}\right)$

35) $\left(8r^{5}+19s^{5}\right)\left(8r^{5}-19s^{5}\right)$

Answer

$64r^{10}-361s^{10}$

36) $\left(3u^{3}+16v^{3}\right)\left(3u^{3}-16v^{3}\right)$

In Exercises 37-60, factor each of the given expressions.

37) $361x^{2}-529$

Answer

$(19x+23)(19x-23)$

38) $9b^{2}-25$

39) $16v^{2}-169$

Answer

$(4v+13)(4v-13)$

40) $81r^{2}-169$

41) $169x^{2}-576y^{2}$

Answer

$(13x+24y)(13x-24y)$

42) $100y^{2}-81z^{2}$

43) $529r^{2}-289s^{2}$

Answer

$(23r+17s)(23r-17s)$

44) $49a^{2}-144b^{2}$

45) $49r^{6}-256t^{6}$

Answer

$\left(7r^{3}+16t^{3}\right)\left(7r^{3}-16t^{3}\right)$

46) $361x^{10}-484z^{10}$

47) $36u^{10}-25w^{10}$

Answer

$\left(6u^{5}+5w^{5}\right)\left(6u^{5}-5w^{5}\right)$

48) $a^{6}-81c^{6}$

49) $72y^{5}-242y^{3}$

Answer

$2y^3(6y+11)(6y-11)$

50) $75y^5-147y^3$

51) $1444a^3b-324ab^3$

Answer

$4ab(19a+9b)(19a-9b)$

52) $12b^3c-1875bc^3$

53) $576x^3z-1156xz^3$

Answer

$4xz(12x+17z)(12x-17z)$

54) $192u^3v-507uv^3$

55) $576t^4-4t^2$

Answer

$4t^2(12t+1)(12t-1)$

56) $4z^5-256z^3$

57) $81x^4-256$

Answer

$\left(9x^2+16\right)(3x+4)(3x-4)$

58) $81x^4-1$

59) $81x^4-16$

Answer

$\left(9x^2+4\right)(3x+2)(3x-2)$

60) x^4-1

In Exercises 61-68, factor each of the given expressions completely.

61) z^3+z^2-9z-9

Answer

$(z+3)(z-3)(z+1)$

62) $3u^3+u^2-48u-16$

63) $x^3-2x^2y-xy^2+2y^3$

Answer

$(x+y)(x-y)(x-2y)$

64) $x^3+2x^2z-4xz^2-8z^3$

65) $r^3-3r^2t-25rt^2+75t^3$

Answer

$(r+5t)(r-5t)(r-3t)$

66) $2b^3 - 3b^2c - 50bc^2 + 75c^3$

67) $2x^3 + x^2 - 32x - 16$

Answer

$(x+4)(x-4)(2x+1)$

68) $r^3 - 2r^2 - r + 2$

In Exercises 69-80, solve each of the given equations for x.

69) $2x^3 + 7x^2 = 72x + 252$

Answer

$$x = -6, 6, -\frac{7}{2}$$

70) $2x^3 + 7x^2 = 32x + 112$

71) $x^3 + 5x^2 = 64x + 320$

Answer

$x = -8, 8, -5$

72) $x^3 + 4x^2 = 49x + 196$

73) $144x^2 + 121 = 264x$

Answer

$$x = \frac{11}{12}$$

74) $361x^2 + 529 = 874x$

75) $16x^2 = 169$

Answer

$$x = -\frac{13}{4}, \frac{13}{4}$$

76) $289x^2 = 4$

77) $9x^2 = 25$

Answer

$$x = -\frac{5}{3}, \frac{5}{3}$$

78) $144x^2 = 121$

79) $256x^2 + 361 = -608x$

Answer

$$x = -\frac{19}{16}$$

80) $16x^2 + 289 = -136x$

In Exercises 81-84, perform each of the following tasks:

a. **Use a strictly algebraic technique to solve the given equation.**
b. **Use the 5:intersect utility on your graphing calculator to solve the given equation.**

Report the results found using graphing calculator as shown in Example 6.5.12.

81) $x^3 = x$

Answer

$x = 0, -1, 1$

82) $x^3 = 9x$

83) $4x^3 = x$

Answer

$x = 0, -\frac{1}{2}, \frac{1}{2}$

84) $9x^3 = x$

6.6: Factoring Strategy

In Exercises 1-12, factor each of the given polynomials completely.

1) $484y^4z^2 - 144y^2z^4$

Answer

$4y^2z^2(11y + 6z)(11y - 6z)$

2) $72s^4t^4 - 242s^2t^6$

3) $3x^7z^5 - 363x^5z^5$

Answer

$3x^5z^5(x + 11)(x - 11)$

4) $5r^5s^2 - 80r^3s^2$

5) $2u^7 - 162u^5$

Answer

$2u^5(u + 9)(u - 9)$

6) $405x^4 - 320x^2$

7) $3v^8 - 1875v^4$

Answer

$3v^4(v^2 + 25)(v + 5)(v - 5)$

8) $3a^9 - 48a^5$

9) $3x^6 - 300x^4$

Answer

$3x^4(x + 10)(x - 10)$

10) $2y^5 - 18y^3$

11) $1250u^7w^3 - 2u^3w^7$

Answer

$2u^3w^3\left(25u^2 + w^2\right)(5u + w)(5u - w)$

12) $48y^8z^4 - 3y^4z^8$

In Exercises 13-24, factor each of the given polynomials completely.

13) $75a^6 - 210a^5 + 147a^4$

Answer

$3a^4(5a-7)^2$

14) $245v^7 - 560v^6 + 320v^5$

15) $180a^5b^3 + 540a^4b^4 + 405a^3b^5$

Answer

$45a^3b^3(2a+3b)^2$

16) $192u^6v^4 + 432u^5v^5 + 243u^4v^6$

17) $2b^5 + 4b^4 + 2b^3$

Answer

$2b^3(b+1)^2$

18) $3v^6 + 30v^5 + 75v^4$

19) $2z^4 - 4z^3 + 2z^2$

Answer

$2z^2(z-1)^2$

20) $2u^6 - 40u^5 + 200u^4$

21) $324x^4 + 360x^3 + 100x^2$

Answer

$4x^2(9x+5)^2$

22) $98b^4 + 84b^3 + 18b^2$

23) $75b^4c^5 - 240b^3c^6 + 192b^2c^7$

Answer

$3b^2c^5(5b-8c)^2$

24) $162a^5c^4 - 180a^4c^5 + 50a^3c^6$

In Exercises 25-36, factor each of the given polynomials completely.

25) $5a^5 + 5a^4 - 210a^3$

Answer

$5a^3(a-6)(a+7)$

26) $3y^5 - 9y^4 - 12y^3$

27) $3y^6 - 39y^5 + 120y^4$

Answer

$3y^4(y-8)(y-5)$

28) $3y^7 - 27y^6 + 42y^5$

29) $3z^4 + 12z^3 - 135z^2$

Answer

$3z^2(z-5)(z+9)$

30) $5a^4 - 40a^3 - 45a^2$

31) $4a^6 + 64a^5 + 252a^4$

Answer

$4a^4(a+9)(a+7)$

32) $4x^4 + 64x^3 + 252x^2$

33) $3z^4 + 33z^3 + 84z^2$

Answer

$3z^2(z+7)(z+4)$

34) $5a^6 + 65a^5 + 180a^4$

35) $5z^7 - 75z^6 + 270z^5$

Answer

$5z^5(z-6)(z-9)$

36) $3y^4 - 27y^3 + 24y^2$

In Exercises 37-48, factor each of the given polynomials completely.

37) $4b^3 - 22b^2 + 30b$

Answer

$2b(2b-5)(b-3)$

38) $4b^6 - 22b^5 + 30b^4$

39) $2u^4w^5 - 3u^3w^6 - 20u^2w^7$

Answer

$u^2w^5(u-4w)(2u+5w)$

40) $12x^5z^2 + 9x^4z^3 - 30x^3z^4$

41) $12x^4y^5 + 50x^3y^6 + 50x^2y^7$

Answer

$2x^2y^5(3x+5y)(2x+5y)$

42) $24s^4t^3 + 62s^3t^4 + 40s^2t^5$

43) $12x^3 + 9x^2 - 30x$

Answer

$3x(4x-5)(x+2)$

44) $6v^4 + 2v^3 - 20v^2$

45) $8u^6 + 34u^5 + 30u^4$

Answer

$2u^4(4u+5)(u+3)$

46) $4a^4+29a^3+30a^2$

47) $12a^4c^4-35a^3c^5+25a^2c^6$

Answer

$a^2c^A(4a-5c)(3a-5c)$

48) $18x^6z^5-39x^5z^6+18x^4z^7$

In Exercises 49-56, factor each of the given polynomials completely.

49) $12y^5+15y^4-108y^3-135y^2$

Answer

$3y^2(y+3)(y-3)(4y+5)$

50) $9b^8+12b^7-324b^6-432b^5$

51) $9x^6z^5+6x^5z^6-144x^4z^7-96x^3z^8$

Answer

$3x^3z^5(x+4z)(x-4z)(3x+2z)$

52) $12u^7w^3+9u^6w^4-432u^5w^5-324u^4w^6$

53) $72z^6+108z^5-2z^4-3z^3$

Answer

$z^3(6z+1)(6z-1)(2z+3)$

54) $216x^7+324x^6-6x^5-9x^4$

55) $144a^6c^3+360a^5c^4-4a^4c^5-10a^3c^6$

Answer

$2a^3c^3(6a+c)(6a-c)(2a+5c)$

56) $48a^8c^4+32a^7c^5-3a^6c^6-2a^5c^7$

In Exercises 57-60, use your calculator to help factor each of the given trinomials. Follow the procedure outline in Using the Calculator to Assist the ac-Method.

57) $6x^2+61x+120$

Answer

$(2x+15)(3x+8)$

58) $16x^2-62x-45$

59) $60x^2-167x+72$

Answer

$(15x-8)(4x-9)$

60) $28x^2+x-144$

6.7: Applications of Factoring

1) A rectangular canvas picture measures 14 inches by 36 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 720 square inches. Find the uniform width of the frame.

Answer

2 inches

2) A rectangular canvas picture measures 10 inches by 32 inches. The canvas is mounted inside a frame of uniform width, increasing the total area covered by both canvas and frame to 504 square inches. Find the uniform width of the frame.

3) A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 180t$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 352t + 1664$$

where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

Answer

26 seconds, $4,680$ feet

4) A projectile is fired at an angle into the air from atop a cliff overlooking the ocean. The projectile's distance (in feet) from the base of the cliff is give by the equation

$$x = 140t$$

and the projectile's height above sea level (in feet) is given by the equation

$$y = -16t^2 + 288t + 1408$$

where t is the amount of time (in seconds) that has passed since the projectile's release. How much time passes before the projectile splashes into the ocean? At that time, how far is the projectile from the base of the cliff?

5) The product of two consecutive even integers is 624. Find the integers.

Answer

-26 and -24, and 24 and 26

6) The product of two consecutive even integers is 528. Find the integers.

7) The product of two consecutive positive integers is 552. Find the integers.

Answer

23, 24

8) The product of two consecutive positive integers is 756. Find the integers.

9) The product of two consecutive odd integers is 483. Find the integers.

Answer

-23 and -21, and 21 and 23

10) The product of two consecutive odd integers is 783. Find the integers.

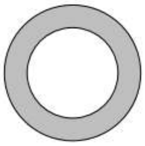
11) A rectangle has perimeter 42 feet and area 104 square feet. Find the dimensions of the rectangle.

Answer

8 feet by 13 feet

12) A rectangle has perimeter 32 feet and area 55 square feet. Find the dimensions of the rectangle.

13) The radius of the outer circle is one inch longer than twice the radius of the inner circle.

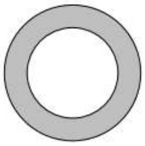


If the area of the shaded region is 40π square inches, what is the length of the inner radius?

Answer

3 inches

14) The radius of the outer circle is two inches longer than three times the radius of the inner circle.



If the area of the shaded region is 180π square inches, what is the length of the inner radius?

15) You have two positive numbers. The second number is three more than two times the first number. The difference of their squares is 144. Find both positive numbers.

Answer

5 and 13

16) You have two positive numbers. The second number is two more than three times the first number. The difference of their squares is 60. Find both positive numbers.

17) Two numbers differ by 5. The sum of their squares is 97. Find the two numbers.

Answer

4 and 9, and -4 and -9

18) Two numbers differ by 6. The sum of their squares is 146. Find the two numbers.

19) The length of a rectangle is three feet longer than six times its width. If the area of the rectangle is 165 square feet, what is the width of the rectangle?

Answer

5 feet

20) The length of a rectangle is three feet longer than nine times its width. If the area of the rectangle is 90 square feet, what is the width of the rectangle?

21) The ratio of the width to the length of a given rectangle is 2 to 3, or $\frac{2}{3}$. If the width and length are both increased by 4 inches, the area of the resulting rectangle is 80 square inches. Find the width and length of the original rectangle.

Answer

4 inches by 6 inches

22) The ratio of the width to the length of a given rectangle is 3 to 4, or $\frac{3}{4}$. If the width is increased by 3 inches and the length is increased by 6 inches, the area of the resulting rectangle is 126 square inches. Find the width and length of the original rectangle.

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CHAPTER OVERVIEW

7: Rational Expressions

7.1: Negative Exponents
7.2: Scientific Notation
7.3: Simplifying Rational Expressions
7.4: Solving Rational Equations
7.5: Direct and Inverse Variation
7.E: Rational Expressions(Exercises)

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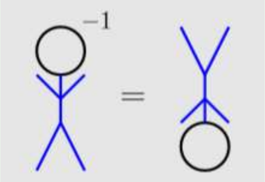
7.1: Negative Exponents

We begin with a seemingly silly but powerful definition on what it means to raise a number to a power of -1.

Raising to a Power of -1

To raise an object to a power of -1, simply invert the object (turn it upside down).



More formally, inverting a number is known as taking its *reciprocal*.

Example 7.1.1

Simplify each of the following expressions:

a. 4^{-1}
b. $\left(\frac{2}{3}\right)^{-1}$
c. $-\left(\frac{3}{5}\right)^{-1}$

Solution

In each case, we simply invert the given number.

a. $\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$
b. $-\left(\frac{3}{5}\right)^{-1}=-\frac{5}{3}$

Exercise 7.1.1

Simplify: $\left(\frac{7}{4}\right)^{-1}$

Answer

$\frac{4}{7}$

You might be asking “Why does raising to the power of minus one invert the number?” To answer this question, recall the product of a number and its reciprocal is one. For example,

$$4\cdot\frac{1}{4}=1 \tag{7.1.1}$$

Next, consider what happens when we multiply 4^1 and 4^{-1}. If we apply the usual law of exponents (assuming they work for both positive and negative exponents), we would add the exponents $(1+(-1)=0)$.

$$4^1\cdot 4^{-1}=4^0 \tag{7.1.2}$$

However, because $4^1=4$ and $4^0=1$, this last equation is equivalent to:

$$4\cdot 4^{-1}=1 \tag{7.1.3}$$

When you compare Equation 7.1.1 and 7.1.3, it is clear that 4^{-1} and $1/4$ are both reciprocals of the number 4. Because reciprocals are unique, $4^{-1}=\frac{1}{4}$.

In similar fashion, one can discover the meaning of a^{-n}. Start with the fact that multiplying reciprocals yields an answer of one.

$$a^n \cdot \frac{1}{a^n} = 1 \tag{7.1.4}$$

If we multiply a^n and a^{-n}, we add the exponents as follows.

$$a^n \cdot a^{-n} = a^0$$

Providing $a \neq= 0$, then $a^0 = 1$, so we can write

$$a^n \cdot a^{-n} = 1 \tag{7.1.5}$$

Comparing Equations 7.1.4 and 7.1.5, we note that both $1/a^n$ and a^{-n} are reciprocals of a^n. Because every number has a unique reciprocal, a^{-n} and $1/a^n$ are equal.

Raising to a negative integer

Provided a\neq= 0,

$$a^{-n} = \frac{1}{a^n}$$

Example 7.1.2

Simplify each of the following expressions:

a. 2^{-3}
b. $(-5)^{-2}$
c. $(-4)^{-3}$

Solution

In each example, we use the property $a^{-n} = 1/a^n$ to simplify the given expression.

a. $$\begin{aligned} 2^{-3} &= \frac{1}{2^3} \\ &= \frac{1}{8} \end{aligned}$$

b. $$\begin{aligned} (-5)^{-2} &= \frac{1}{(-5)^2} \\ &= \frac{1}{25} \end{aligned}$$

c. $$\begin{aligned} (-4)^{-3} &= \frac{1}{(-4)^3} \\ &= -\frac{1}{64} \end{aligned}$$

In Raising to a Negative Integer, we'll address how you can perform each of the above computations mentally.

Exercise 7.1.2

Simplify: 3^{-2}

Answer

$$\frac{1}{9}$$

Laws of Exponents

In the arguments demonstrating that $4^{-1} = 1/4$ and $a^{-n} = 1/a^n$, we appealed to one of the laws of exponents learned in Chapter 5, Section 5. Fortunately, the laws of exponents work exactly the same whether the exponents are positive or negative integers.

Laws of Exponents

If m and n are integers, then:

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{a^n}$

Example 7.1.3

Simplify each of the following expressions:

a. $y^5 y^{-7}$
b. $2^{-2} \cdot 2^{-3}$
c. $x^{-4} x^6$

Solution

In each case, we use the first law of exponents ($a^m a^n = a^{m+n}$). Because we are multiplying like bases, we repeat the base and add the exponents.

a. $$\begin{aligned} y^5 y^{-7} &= y^{5+(-7)} \\ &= y^{-2} \end{aligned}$$
b. $$\begin{aligned} 2^{-2} \cdot 2^{-3} &= 2^{-2+(-3)} \\ &= 2^{-5} \end{aligned}$$
c. $$\begin{aligned} x^{-4} x^6 &= x^{-4+6} \\ &= x^2 \end{aligned}$$

Exercise 7.1.3

Simplify: $t^8 \cdot t^{-4}$

Answer

t^4

Example 7.1.4

Simplify each of the following expressions:

1. $\frac{x^4}{x^7}$
2. $\frac{3^{-4}}{3^5}$
3. $\frac{z^{-3}}{z^{-5}}$

Solution

In each case, we use the second law of exponents ($a^m / a^n = a^{m-n}$). Because we are dividing like bases, we repeat the base and subtract the exponents. Recall that subtraction means “add the opposite.”

a. $$\begin{aligned} \frac{x^4}{x^7} &= x^{4-7} \\ &= x^{4+(-7)} \\ &= x^{-3} \end{aligned}$$
b. $$\begin{aligned} \frac{3^{-4}}{3^5} &= 3^{-4-5} \\ &= 3^{-4+(-5)} \\ &= 3^{-9} \end{aligned}$$

c. $$\begin{aligned}\frac{z^{-3}}{z^{-5}} &= z^{-3-(-5)} \\ &= z^{-3+5} \\ &= z^2\end{aligned}$$

Exercise 7.1.4

Simplify: $\frac{y^{-6}}{y^{-2}}$

Answer

y^{-4}

Example 7.1.5

Simplify each of the following expressions:

a. $(5^{-2})^3$
b. $(a^{-3})^{-4}$
c. $(w^2)^{-7}$

Solution

In each case, we are using the third law of exponents ($(a^m)^n = a^{mn}$). Because we are raising a power to another power, we repeat the base and multiply the exponents.

a. $$\begin{aligned}(5^{-2})^3 &= 5^{(-2)(3)} \\ &= 5^{-6}\end{aligned}$$
b. $$\begin{aligned}(a^{-3})^{-4} &= a^{(-3)(-4)} \\ &= a^{12}\end{aligned}$$
c. $$\begin{aligned}(w^2)^{-7} &= w^{(2)(-7)} \\ &= w^{-14}\end{aligned}$$

Exercise 7.1.5

Simplify: $(z^5)^{-2}$

Answer

\(z^{-10)\)

Raising to a Negative Integer

We know what happens when you raise a number to -1, you invert the number or turn it upside down. But what happens when you raise a number to a negative integer other than negative one?

As an example, consider the expression 3^{-2}. Using the third law of exponents ($(a^m)^n = a^{mn}$), we can write this expression in two equivalent forms.

1. Note that 3^{-2} is equivalent to $(3^2)^{-1}$. They are equivalent because the third law of exponents instructs us to multiply the exponents when raising a power to another power. Finally, note that to evaluate $(3^2)^{-1}$, we first square, then invert the result.

$$\begin{aligned}3^{-2} &= (3^2)^{-1} && \text{Repeat base and multiply exponents.} \\ &= 9^{-1} && \text{Simplify: } 3^2 = 9 \\ &= \frac{1}{9} && \text{Simplify: } 9^{-1} = 1/9\end{aligned}$$

2. Note that 3^{-2} is also equivalent to $(3^{-1})^2$. They are equivalent because the third law of exponents instructs us to multiply the exponents when raising a power to another power. Finally, note that to evaluate $(3^{-1})^2$, we first invert, then square the result.

$$\begin{aligned} 3^{-2} &= (3^{-1})^2 && \text{Repeat base and multiply exponents.} \\ &= \left(\frac{1}{3}\right)^2 && \text{Simplify: } 3^{-1} = 1/3 \\ &= \frac{1}{9} && \text{Simplify: } (1/3)^2 = 1/9 \end{aligned}$$

Using either technique, $3^{-2} = 1/9$. You can either square and invert, or you can invert and square. In each case, the 2 means "square" and the minus sign means "invert," and this example shows that it doesn't matter which you do first.

Example 7.1.6

Simplify each of the following expressions:

a. 5^{-3}
b. $(-4)^{-2}$
c. $\left(\frac{3}{5}\right)^{-2}$
d. $\left(-\frac{2}{3}\right)^{-3}$

Solution

1. We'll cube then invert.

$$\begin{aligned} 5^{-3} &= (5^3)^{-1} && \text{Repeat base and multiply exponents.} \\ &= 125^{-1} && \text{Simplify: } 5^3 = 125 \\ &= \frac{1}{125} && \text{Invert: } 125^{-1} = 1/125 \end{aligned}$$

Note that the three means "cube" and the minus sign means "invert," so it is possible to do all of this work mentally: cube 5 to get 125, then invert to get $1/125$.

2. We'll square then invert.

$$\begin{aligned} (-4)^{-2} &= ((-4)^2)^{-1} && \text{Repeat base and multiply exponents.} \\ &= 16^{-1} && \text{Simplify: } (-4)^2 = 16 \\ &= \frac{1}{16} && \text{Invert: } 16^{-1} = 1/16 \end{aligned}$$

Note that the two means "square" and the minus sign means "invert," so it is possible to do all of this work mentally: square -4 to get 16, then invert to get $1/16$.

3. Again, we'll square then invert.

$$\begin{aligned} \left(\frac{3}{5}\right)^{-2} &= \left(\left(\frac{3}{5}\right)^2\right)^{-1} && \text{Repeat base and multiply exponents.} \\ &= \left(\frac{9}{25}\right)^{-1} && \text{Simplify: } (3/5)^2 = 9/25 \\ &= \frac{25}{9} && \text{Invert: } (9/25)^{-1} = 25/9 \end{aligned}$$

Note that the two means "square" and the minus sign means "invert," so it is possible to do all of this work mentally: square $3/5$ to get $9/25$, then invert to get $25/9$.

4. This time we'll cube then invert.

$$\begin{aligned} \left(-\frac{2}{3}\right)^{-3} &= \left(\left(-\frac{2}{3}\right)^3\right)^{-1} && \text{Repeat base and multiply exponents.} \\ &= \left(-\frac{8}{27}\right)^{-1} && \text{Simplify: } (-2/3)^2 = -8/27 \\ &= -\frac{27}{8} && \text{Invert: } (-8/27)^{-1} = -27/8 \end{aligned}$$

Note that the three means "cube" and the minus sign means "invert," so it is possible to do all of this work mentally: cube $-2/3$ to get $-8/27$, then invert to get $-27/8$.

Exercise 7.1.6

Simplify: $\left(\frac{5}{4}\right)^{-3}$

Answer

$\frac{64}{125}$

Applying the Laws of Exponents

In this section we'll simplify a few more complicated expressions using the laws of exponents.

Example 7.1.7

Simplify: $(2x^{-2}y^{3})(-3x^{5}y^{-6})$

Solution

All the operators involved are multiplication, so the commutative and associative properties of multiplication allow us to change the order and grouping. We'll show this regrouping here, but this step can be done mentally.

$$(2x-2y^{3})(-3x^{5}y^{-6})=[(2)(-3)](x^{-2}x^{5})(y^{3}y^{-6})$$

When multiplying, we repeat the base and add the exponents.

$$\begin{aligned}&=-6x^{-2+5}y^{3+(-6)}\\&=-6x^{3}y^{-3}\end{aligned}$$

In the solution above, we've probably shown way too much work. It's far easier to perform all of these steps mentally, multiplying the 2 and the -3, then repeating bases and adding exponents, as in:

$$(2x^{-2}y^{3})(-3x^{5}y^{-6})=-6x^{3}y^{-3}$$

Exercise 7.1.7

Simplify: $(-5x^{8}y^{-2})(-2x^{-6}y^{-1})$

Answer

$10x^{2}y^{-3}$

Example 7.1.8

Simplify: $\frac{6x^{-2}y^{5}}{9x^{3}y^{-2}}$

Solution

The simplest approach is to first write the expression as a product.

$$\frac{6x^{-2}y^{5}}{9x^{3}y^{-2}}=\frac{6}{9}\cdot\frac{x^{-2}}{x^{3}}\cdot\frac{y^{5}}{y^{-2}}$$

Reduce $6/9$ to lowest terms. Because we are dividing like bases, we repeat the base and subtract the exponents.

$$\begin{aligned}&=\frac{2}{3}x^{-2-3}y^{5-(-2)}\\&=\frac{2}{3}x^{-2+(-3)}y^{5+2}\\&=\frac{2}{3}x^{-5}y^{7}\end{aligned}$$

In the solution above, we've probably shown way too much work. It's far easier to imagine writing the expression as a product, reducing 6/9, then repeating bases and subtracting exponents, as in:

$$\frac{6x^{-2}y^5}{9x^3y^{-2}} = \frac{2}{3}x^{-5}y^7$$

Exercise 7.1.8

Simplify: $\frac{10x^3y^{-1}}{4x^{-2}y^5}$

Answer

$\frac{5}{2}x^5y^{-6}$

Example 7.1.9

Simplify: $(2x^{-2}y^4)^{-3}$

Solution

The fourth law of exponents ($(ab)^n = a^nb^n$) says that when you raise a product to a power, you must raise each factor to that power. So we begin by raising each factor to the minus three power.

$$(2x^{-2}y^3)^{-3} = 2^{-3}(x^{-2})^{-3}(y^4)^{-3}$$

To raise two to the minus three, we must cube two and invert: $2^{-3} = 1/8$. Secondly, raising a power to a power requires that we repeat the base and multiply exponents.

$$\begin{aligned} &= \frac{1}{8}x^{(-2)(-3)}y^{(4)(-3)} \\ &= \frac{1}{8}x^6y^{-12} \end{aligned}$$

In the solution above, we've probably shown way too much work. It's far easier raise each factor to the minus three mentally: $2^{-3} = 1/8$, then multiply each exponent on the remaining factors by -3, as in

$$(2x^{-2}y^4)^{-3} = \frac{1}{8}x^6y^{-12}$$

Exercise 7.1.9

Simplify: $(3x^4y^{-3})^{-2}$

Answer

$\frac{1}{9}x^{-8}y^6$

Clearing Negative Exponents

Often, we're asked to provide a final answer that is free of negative exponents. It is common to hear the instruction "no negative exponents in the final answer." Let's explore a couple of techniques that allow us to clear our answer of negative exponents.

Example 7.1.10

Consider the expression:

$$\frac{x^2}{y^{-3}}$$

Simplify so that the resulting equivalent expression contains no negative exponents.

Solution

Raising y to the -3 means we have to cube and invert, so $y^{-3} = 1/y^3$.

$$\frac{x^2}{} = \frac{x^2}{}$$

$$\overline{y^{-3}} = \overline{\frac{1}{y^3}}$$

To divide x^2 by $1/y^3$, we invert and multiply.

$$\begin{aligned} &= x^2 \div \frac{1}{y^3} \\ &= \frac{x^2}{1} \cdot \frac{y^3}{1} \\ &= x^2 y^3 \end{aligned}$$

Alternate approach: An alternate approach takes advantage of the laws of exponents. We begin by multiplying numerator and denominator by y^3.

$$\begin{aligned} \frac{x^2}{y^{-3}} &= \frac{x^2}{y^{-3}} \cdot \frac{y^3}{y^3} \\ &= \frac{x^2 y^3}{y^0} \\ &= x^2 y^3 \end{aligned}$$

In the last step, note how we used the fact that $y^0 = 1$

Exercise 7.1.10

Simplify the expression

$$\frac{y^5}{x^{-2}}$$

so that the resulting equivalent expression contains no negative exponents.

Answer

$y^5 x^2$

Example 7.1.11

Consider the expression:

$$\frac{2x^2 y^{-2}}{z^3}$$

Simplify so that the resulting equivalent expression contains no negative exponents.

Solution

Again, we can remove all the negative exponents by taking reciprocals. In this case $y^{-2} = 1/y^2$ (square and invert).

$$\begin{aligned} \frac{2x^2 y^{-2}}{z^3} &= \frac{2x^2 \cdot \frac{1}{y^2}}{z^3} \\ &= \frac{\frac{2x^2}{y^2}}{z^3} \end{aligned}$$

To divide $2x^2/y^2$ by z^3, we invert and multiply.

$$\begin{aligned} &= \frac{2x^2}{y^2} \div z^3 \\ &= \frac{2x^2}{y^2} \cdot \frac{1}{z^3} \\ &= \frac{2x^2}{y^2 z^3} \end{aligned}$$

Alternate approach: An alternate approach again takes advantage of the laws of exponents. We begin by multiplying numerator and denominator by y^2.

$$\begin{aligned}\frac{2x^2y^{-2}}{z^3} &= \frac{2x^2y^{-2}}{z^3}\cdot\frac{y^2}{y^2}\\ &= \frac{2x^2y^0}{y^2z^3}\\ &= \frac{2x^2}{y^2z^3}\end{aligned}$$

In the last step, note how we used the fact that $y^0 = 1$.

Exercise 7.1.11

Simplify the expression

$$\frac{x^{-3}y^2}{3z^{-4}}$$

so that the resulting equivalent expression contains no negative exponents.

Answer

$\dfrac{y^2z^4}{3x^3}$

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7.2: Scientific Notation

We begin this section by examining powers of ten.

$$\begin{aligned}10^1 &= 10\\10^2 &= 10\cdot 10 = 100\\10^3 &= 10\cdot 10\cdot 10 = 1,000\\10^4 &= 10\cdot 10\cdot 10\cdot 10 = 10,000\end{aligned}$$

Note that the answer for 10^3 is a one followed by three zeros. The answer for 10^4 is a one followed by four zeros. Do you see the pattern?

Nonnegative powers of ten

In the expression 10^n, the exponent matches the number of zeros in the answer. Hence, 10^n will be a 1 followed by n zeros.

Example 7.2.1

Simplify: 10^9.

Solution

10^9 should be a 1 followed by 9 zeros.

$$10^9 = 1,000,000,000$$

Exercise 7.2.1

Simplify: 10^6.

Answer

$1,000,000$

Next, let's examine negative powers of ten.

$$\begin{aligned}10^{-1} &= \frac{1}{10} = 0.1\\10^{-2} &= \frac{1}{100} = 0.01\\10^{-3} &= \frac{1}{1000} = 0.001\\10^{-4} &= \frac{1}{10000} = 0.0001\end{aligned}$$

Note that the answer for 10^{-3} has three decimal places and the answer for 10^{-4} contains four decimal places.

Negative powers of ten

In the expression 10^{-n}, the exponent matches the number of decimal places in the answer. Hence, 10^{-n} will have n decimal places, the first $n-1$ of which are zeros and the digit in the nth decimal place is a 1.

Example 7.2.2

Simplify: 10^{-7}.

Solution

10^{-7} should have seven decimal places, the first six of which are zeros, and the digit in the seventh decimal place is a 1.

$$10^{-7} = 0.0000001$$

Exercise 7.2.2

Simplify: 10^{-5}.

Answer

0.00001

Multiplying Decimal Numbers by Powers of Ten

Let's multiply 1.234567 by 10^3, or equivalently, by $1,000$.

$$\begin{array}{r} 1.234567 \\ \times 1000 \\ \hline 1234.567000 \end{array}$$

The sum total of digits to the right of the decimal point in 1.234567 and 1000 is 6. Therefore, we place the decimal point in the product so that there are six digits to the right of the decimal point.

However, the trailing zeros may be removed without changing the value of the product. That is, 1.234567 times 1000 is 1234.567. Note that the decimal point in the product is three places further to the right than in the original factor. This observation leads to the following result.

Multiplying by a nonnegative power of ten

Multiplying a decimal number by 10^n, where $n = 0, 1, 2, 3, \ldots$, will move the decimal point n places to the right.

Example 7.2.3

Simplify: 325.6783×10^2.

Solution

Multiplying by 10^2 will move the decimal point two places to the right. Thus:

$$325.6783 \times 10^2 = 32,567.83$$

Exercise 7.2.3

Simplify: 23.57889×10^3

Answer

$23,578.89$

Example 7.2.4

Simplify: 1.25×10^5.

Solution

Multiplying by 10^5 will move the decimal point two places to the right. In this case, we need to add zeros at the end of the number to accomplish moving the decimal 5 places to the right.

$$1.25 \times 105 = 125,000$$

Exercise 7.2.14

Simplify: 2.35×10^4

Answer

$23,500$

Let's multiply 453.9 by 10^{-2}, or equivalently, by 0.01.

$$\begin{array}{r} 453.9 \\ \times 0.01 \\ \hline 4.539 \end{array}$$

The sum total of digits to the right of the decimal point in 453.9 and 0.01 is 3. Therefore, we place the decimal point in the product so that there are 3 digits to the right of the decimal point. That is, $453.9 \times 10^{-2} = 4.539$. Note that the decimal point in the product is two places further to the left than in the original factor. This observation leads to the following result.

Multiplying by a negative power of ten

Multiplying a decimal number by 10^{-n}, where $n = 1, 2, 3, \ldots$, will move the decimal point n places to the left.

Example $7.2.5$

Simplify: $14,567.8 \times 10^{-3}$.

Solution

Multiplying by 10^{-3} will move the decimal point three places to the left. Thus:

$$14,567.8 \times 10^{-3} = 14.5678$$

Exercise $7.2.5$

Simplify: $3,854.2 \times 10^{-1}$

Answer

385.42

Example $7.2.6$

Simplify: 4.3×10^{-4}.

Solution

Multiplying by 10^{-4} will move the decimal point four places to the left. In this case, we need to add some leading zeros at the beginning of the number to accomplish moving the decimal 4 places to the left.

$$4.3 \times 10^{-4} = 0.00043$$

. Note also the leading zero before the decimal point. Although $.00043$ is an equivalent number, the form 0.00043 is preferred in mathematics and science.

Exercise $7.2.6$

Simplify: 2.2×10^{-2}

Answer

0.022

Scientific Notation Form

We start by defining the form of a number called *scientific notation*.

Scientific notation

A number having the form

$$a \times 10^b$$

where b is an integer and $1 \leq |a| < 10$, is said to be in *scientific notation*.

The requirement $1 \leq |a| < 10$ says that the magnitude of a must be a least 1 and less than 10.

- The number 12.34×10^{-4} is **not** in scientific notation because $|12.34| = 12.34$ is larger than 10.
- The number -0.95×10^3 is **not** in scientific notation because $|-0.95| = 0.95$ is less than 1.
- The number 7.58×10^{-12} **is** in scientific notation because $|7.58| = 7.58$ is greater than or equal to 1 and less than 10.
- The number -1.0×10^{15} **is** in scientific notation because $|-1.0| = 1.0$ is greater than or equal to 1 and less than 10.

After contemplating these examples, it follows that a number in scientific notation should have exactly one of the digits $1, 2, 3, \ldots, 9$ before the decimal point. Exactly one, no more, no less. Thus, each of the following numbers is in scientific notation.

$$4.7 \times 10^8, \quad -3.764 \times 10^{-1}, \quad 3.2 \times 10^0, \quad \text{and} \quad -1.25 \times 10^{-22}$$

Placing a Number in Scientific Notation

To place a number into scientific notation, we need to move the decimal point so that exactly one of the digits $1, 2, 3, \ldots, 9$ remains to the left of the decimal point, then multiply by the appropriate power of 10 so that the result is equivalent to the original number.

Example 7.2.7

Place the number $1,234$ in scientific notation.

Solution

Move the decimal point three places to the left so that it is positioned just after the 1. To make this new number equal to $1,234$, multiply by 10^3. Thus:

$$1,234 = 1.234 \times 10^3$$

Check: Multiplying by 10^3 moves the decimal three places to the right, so:

$$1.234 \times 10^3 = 1,234$$

This is the original number, so our scientific notation form is correct.

Exercise 7.2.7

Place the number $54,321$ in scientific notation.

Answer

5.4321×10^4

Example 7.2.8

Place the number 0.000025 in scientific notation.

Solution

Move the decimal point five places to the right so that it is positioned just after the 2. To make this new number equal to 0.000025, multiply by 10^{-5}. Thus:

$$0.000025 = 2.5 \times 10^{-5}$$

Check: Multiplying by 10^{-5} moves the decimal five places to the left, so:

$$2.5 \times 10^{-5} = 0.000025$$

This is the original number, so our scientific notation form is correct.

Exercise 7.2.8

Place the number 0.0175 in scientific notation.

Answer

1.75×10^{-2}

Example 7.2.9

Place the number 34.5×10^{-11} in scientific notation.

Solution

First, move the decimal point one place to the left so that it is positioned just after the three. To make this new form equal to 34.5, multiply by 10^{1}.

$$34.5 \times 10^{-11} = 3.45 \times 10^{1} \times 10^{-11}$$

Now, repeat the base 10 and add the exponents.

$$= 3.45 \times 10^{-10}$$

Exercise 7.2.9

Place the number 756.98×10^{-5} in scientific notation.

Answer

7.5698×10^{-3}

Example 7.2.10

Place the number 0.00093×10^{12} in scientific notation.

Solution

First, move the decimal point four places to the right so that it is positioned just after the nine. To make this new form equal to 0.00093, multiply by 10^{-4}.

$$0.00093 \times 10^{12} = 9.3 \times 10^{-4} \times 10^{12}$$

Now, repeat the base 10 and add the exponents.

$$= 9.3 \times 10^{8}$$

Exercise 7.2.10

Place the number 0.00824×10^{8} in scientific notation.

Answer

8.24×10^{5}

Scientific Notation and the Graphing Calculator

The TI-84 graphing calculator has a special button for entering numbers in scientific notation. Locate the “comma” key just about the number 7 key on the calculator’s keyboard (see Figure $7.2.1$). Just above the “comma” key, printed on the calculator’s case is the symbol **EE**. It’s in the same color as the **2nd** key, so you’ll have to use the **2nd** key to access this symbol.

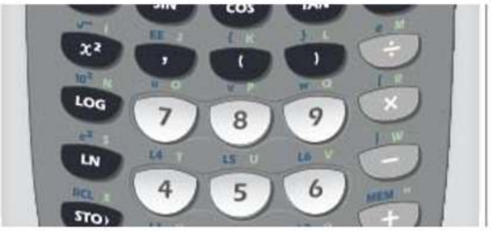
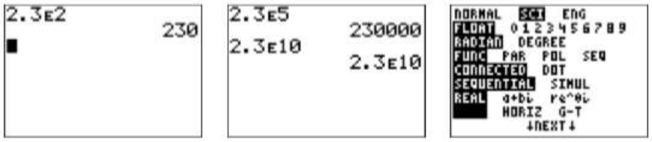


Figure $7.2.1$: Locate the **EE** label above the “comma” key on the keyboard.

We know that $2.3 \times 10^2 = 230$. Let’s see if the calculator gives the same interpretation.

1. Enter 2.3.
2. Press the **2nd** key, then the comma key. This will put **E** on the calculator view screen.
3. Enter a 2.
4. Press **ENTER**.



```
2.3E2
                230
```

```
2.3E5
             230000
2.3E10
             2.3E10
```

```
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNC PAR POL SEQ
CONNECTED DOT
SEQUENTIAL SIMUL
REAL a+bi re^θi
HORIZ G-T
↓NEXT↓
```

Figure $7.2.2$: Entering numbers in scientific notation.

The result of these steps is shown in the first image in Figure $7.2.2$. Note that the calculator interprets $2.3\mathbf{E}2$ as 2.3×10^2 and gives the correct answer, 230. You can continue entering numbers in scientific notation (see the middle image in Figure $7.2.2$). However, at some point the numbers become too large and the calculator responds by outputting the numbers in scientific notaiton. You can also force your calculator to display numbers in scientific notation in all situations, by first pressing the **MODE** key, then selecting **SCI** on the first line and pressing the **ENTER** key (see the third image in Figure $7.2.2$). You can return your calculator to “normal” mode by selecting **NORMAL** and pressing the **ENTER** key.

Example $7.2.11$

Use the graphing calculator to simplify:

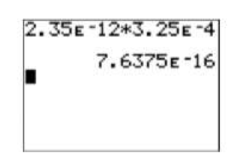
$$(2.35 \times 10^{-12})(3.25 \times 10^{-4})$$

Solution

First, note that we can approximate $(2.35 \times 10^{-12})(3.25 \times 10^{-4})$ by taking the product of 2 and 3 and adding the powers of ten.

$$\begin{aligned} &(2.35 \times 10^{-12})(3.25 \times 10^{-4}) \\ &\approx (2 \times 10^{-12})(3 \times 10^{-4}) \quad \text{Approximate: } 2.35 \approx 2 \text{ and } 3.25 \approx 3 \\ &\approx 6 \times 10^{-16} \quad 2 \cdot 3 = 6 \text{ and } 10^{-12} \cdot 10^{-4} = 10^{-16} \end{aligned}$$

The graphing calculator will provide an accurate answer. Enter $2.35\mathbf{E}-12$, press the “times” button, then enter 3.25E-4 and press the ENTER button. Be sure to use the “negate” button and not the “subtract” button to produce the minus sign. The result is shown in Figure $7.2.3$.



```
2.35E-12*3.25E-4
         7.6375E-16
```

Figure $7.2.3$: Computing $(2.35 \times 10^{-12})(3.25 \times 10^{-4})$.

Thus, $(2.35 \times 10^{-12})(3.25 \times 10^{-4}) = 7.6375 \times 10^{-16}$. Note that this is fairly close to our estimate of 6×10^{-16}.

Exercise 7.2.11

Use the graphing calculator to simplify:

$$(3.42 \times 10^{6})(5.86 \times 10^{-9})$$

Answer

2.00412×10^{-2}

Reporting your answer on your homework

After computing the answer to Example 7.2.11 on your calculator, write the following on your homework:

$$(2.35 \times 10^{-12})(3.25 \times 10^{-4}) = 7.6375 \times 10^{-16}$$

Do not write $7.6375\mathbf{E}-16$.

Example 7.2.12

Use the graphing calculator to simplify:

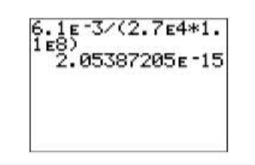
$$\frac{6.1 \times 10^{-3}}{(2.7 \times 10^{4})(1.1 \times 10^{8})}$$

Solution

Again, it is not difficult to produce an approximate answer.

$$\begin{aligned}
&\frac{6.1 \times 10^{-3}}{(2.7 \times 10^{4})(1.1 \times 10^{8})} \\
&\approx \frac{6 \times 10^{-3}}{(3 \times 10^{4})(1 \times 10^{8})} && 6.1 \approx 6, 2.7 \approx 3, \text{ and } 1.1 \approx 1 \\
&\approx \frac{6 \times 10^{-3}}{3 \times 10^{12}} && 3 \cdot 1 = 3 \text{ and } 10^{4} \cdot 10^{8} = 10^{12} \\
&\approx \frac{6}{3} \cdot \frac{10^{-3}}{10^{12}} && \frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d} \\
&\approx 2 \times 10^{-15} && \frac{6}{3} = 2 \text{ and } \frac{10^{-3}}{10^{12}} = 10^{-15}
\end{aligned}$$

Let's get a precise answer with our calculator. Enter the numerator as $6.1\mathbf{E}3$, then press the "division" button. Remember that we must surround the denominator with parentheses. So press the open parentheses key, then enter $2.7\mathbf{E}4$. Press the "times" key, then enter $1.1\mathbf{E}8$. Press the close parentheses key and press the **ENTER** button. The result is shown in Figure 7.2.4.



```
6.1E-3/(2.7E4*1.
1E8)
    2.05387205E-15
```

Figure 7.2.4: Computing $6.1 \times 10^{-3}/(2.7 \times 10^{4} \times 1.1 \times 10^{8})$.

Thus, $6.1 \times 10^{-3}/(2.7 \times 10^{4} \times 1.1 \times 10^{8}) = 2.05387205 \times 10^{-15}$. Note that this is fairly close to our estimate of 2×10^{-15}.

Exercise 7.2.12

Use the graphing calculator to simplify:

$$\frac{2.6 \times 10^{4}}{(7.1 \times 10^{-2})(6.3 \times 10^{7})}$$

Answer

5.8126537×10^{-3}

Example 7.2.13

Isaac Newton's universal law of gravitation is defined by the formula

$$F=\frac{GmM}{r^2}$$

where F is the force of attraction between two objects having mass m and M, r is the distance between the two objects, and G is Newton's gravitational constant defined by:

$$G=6.67428\times 10^{-11}\mathrm{N}(\mathrm{m}/\mathrm{kg})^2$$

Given that the mass of the moon is 7.3477×10^{22} kilograms (kg), the mass of the earth is 5.9736×10^{24} kilograms (kg), and the average distance between the moon and the earth is 3.84403×10^{8} meters (m), find the force of attraction between the earth and the moon (in newtons (N)).

Solution

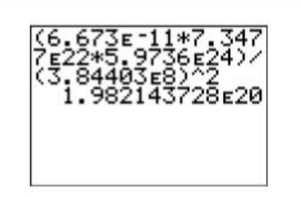
Plug the given numbers into Newton's universal law of gravitation.

$$F=\frac{GmM}{r^2}$$

$$F=\frac{(6.673\times 10^{-11})(7.3477\times 10^{22})(5.9736\times 10^{24})}{(3.84403\times 10^{8})^2}$$

Enter the expression into your calculator (see Figure $7.2.5$) as:

$$(6.673\mathbf{E}-11*7.3477\mathbf{E}22*5.9736\mathbf{E}24)/(3.84403\mathbf{E}8)\wedge 2$$



```
(6.673E-11*7.347
7E22*5.9736E24)/
(3.84403E8)^2
   1.982143728E20
```

Figure $7.2.5$: Computing force of attraction between earth and the moon.

Hence, the force of attraction between the earth and the moon is approximately 1.98×10^{20} newtons (N).

Exercise 7.2.13

The mass of the International Space Station is $450,000$ kg, and its average distance to the center of the earth is $387,000$ m. Find the force of attraction between the earth and the station (in newtons (N)).

Answer

$\approx 1.20\times 109N$

Example 7.2.14

The closest star to the earth is Alpha Centauri, 4.37 light-years from the earth. A light-year is the distance that light will travel in one-year's time. The speed of light is $186,000$ miles per second. How many miles from the earth is Alpha Centauri?

Solution

Because the speed of light is measured in miles per second, let's first compute the number of seconds in 4.37 years. Because there are 365 days in a year, 24 hours in a day, 60 minutes in an hour, and 60 seconds in a minute, we can write:

$$4.37\mathrm{yr}=4.37\mathrm{yr}\times 365\frac{\mathrm{day}}{\mathrm{yr}}\times 24\frac{\mathrm{hr}}{\mathrm{day}}\times 60\frac{\mathrm{min}}{\mathrm{hr}}\times 60\frac{\mathrm{s}}{\mathrm{min}}$$

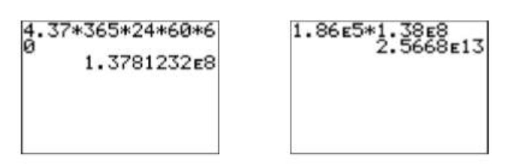
$$= 4.37\,\cancel{\text{yr}} \times 365\frac{}{\cancel{\text{yr}}} \times 24\frac{}{\cancel{\text{day}}} \times 60\frac{}{\cancel{\text{hr}}} \times 60\frac{}{\cancel{\text{min}}}$$

Note how the units cancel, indicating that the final answer is in seconds. With our calculator mode set to scientific notation (see the image on the right in Figure 7.2.2), we multiply the numbers to get the result shown in Figure 7.2.6. Rounding, the number of seconds in 4.37 years is approximately 1.38×10^8 seconds.

Next, we compute the distance the light travels in 4.37 years. Using the fact that the distance traveled equals the speed times the time traveled, we have:

$$\begin{aligned}\text{Distance} &= \text{Speed} \times \text{Time}\\ &= 1.86 \times 10^5 \frac{\text{mi}}{\text{s}} \cdot 1.38 \times 10^8 \text{s}\\ &= 1.86 \times 10^5 \frac{\text{mi}}{\cancel{\text{s}}} \cdot 1.38 \times 10^8 \cancel{\text{s}}\end{aligned}$$

Note how the units cancel, indicating that our answer is in miles. Again, with our calculator set in scientific notation mode, we compute the product of 1.86×10^5 and 1.38×10^8. The result is shown in the image on the right in Figure 7.2.6.



```
4.37*365*24*60*6
0
        1.3781232E8
```

```
1.86E5*1.38E8
         2.5668E13
```

Figure 7.2.6: Computing the distance to Alpha Centauri in miles.

Thus, the star Alpha Centauri is approximately 2.5668×10^{13} miles from the earth, or

$$2.5668 \times 10^{13}\text{miles} \approx 25,668,000,000,000\text{miles}$$

pronounced “twenty-five quadrillion, six hundred sixty-eight trillion miles.”

Exercise 7.2.14

The star Sirius is 8.58 light-years from the earth. How many miles from the earth is Sirius?

Answer

$\approx 5.2425 \times 10^{13}$ miles

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7.3: Simplifying Rational Expressions

Any time you divide a polynomial by a second polynomial, you form what is You will find that material quite helpful for this section. known as a rational expression.

Note

Readers are strongly encouraged to review the material on fractions presented in Section 3 of Chapter 1.

Rational expression

The expression

$$\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials, is called a rational expression.

For example, each of the following is a rational expression.

a. $\dfrac{x+2}{3x}$
b. $\dfrac{x+3}{x^2-2x-4}$
c. $\dfrac{2x}{3y^2}$

In example a), the rational expression is composed of a binomial over a monomial. Example b) is constructed by dividing a binomial by a trinomial. Example c) is composed of a monomial over a monomial, the type of rational expression that will gain the most attention in this section.

Multiplying and Dividing Rational Expressions

We will concentrate on rational expressions with monomial numerators and denominators. Recall that to form the product of two rational numbers, we simply multiply numerators and denominators. The same technique is used to multiply any two rational expressions.

Multiplying rational expressions

Given a/b and c/d, their product is defined as:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Remember, you need only multiply numerators and denominators. For example:

a. $\dfrac{x}{3} \cdot \dfrac{2}{y} = \dfrac{2x}{3y}$
b. $\dfrac{2a}{3b^2} \cdot \dfrac{5a}{9b^3} = \dfrac{10a^2}{27b^5}$
c. $\dfrac{x}{2y} \cdot \left(-\dfrac{3x}{4y^2}\right) = -\dfrac{3x^2}{8y^3}$

Of course, as the next example shows, sometimes you also need to reduce your answer to lowest terms.

Example $7.3.1$

Simplify: $\dfrac{2}{x} \cdot \dfrac{x^2}{4}$.

Solution

Multiply numerators and denominators.

$$\frac{2}{x} \cdot \frac{x^2}{4} = \frac{2x^5}{4x^3}$$

Now, there several different ways you can reduce this answer to lowest terms, two of which are shown below.

You can factor numerator and denominator, then cancel common factors.

$$\begin{aligned}\frac{2x^5}{4x^3} &= \frac{2\cdot x\cdot x\cdot x\cdot x\cdot x}{2\cdot 2\cdot x\cdot x\cdot x}\\ &= \frac{\cancel{2}\cdot \cancel{x}\cdot \cancel{x}\cdot \cancel{x}\cdot x\cdot x}{\cancel{2}\cdot 2\cdot \cancel{x}\cdot \cancel{x}\cdot \cancel{x}}\\ &= \frac{x^2}{2}\end{aligned}$$

Or you can write the answer as a product, repeat the base and subtract exponents.

$$\begin{aligned}\frac{2x^5}{4x^3} &= \frac{2}{4}\cdot\frac{x^5}{x^3}\\ &= \frac{1}{2}\cdot x^{5-3}\\ &= \frac{1}{2}x^2\end{aligned}$$

As dividing by 2 is the same as multiplying by $1/2$, these answers are equivalent. Also, note that the right-hand method is more efficient

Exercise 7.3.1

Simply: $\frac{9}{x^2}\cdot\frac{x}{6}$.

Answer

$\frac{3}{2x}$

Recall that when dividing fractions, we invert the second fraction and multiply.

Dividing rational expressions

Given a/b and c/d, their quotient is defined as:

$$\begin{aligned}\frac{a}{b}\div\frac{c}{d} &= \frac{a}{b}\cdot\frac{d}{c}\\ &= \frac{ad}{bc}\end{aligned}$$

Example 7.3.2

Simplify: $\frac{x^2}{y}\div\frac{x^4}{2y^2}$.

Solution

Invert, then multiply.

$$\begin{aligned}\frac{x^2}{y}\div\frac{x^4}{2y^2} &= \frac{x^2}{y}\cdot\frac{2y^2}{x^4}\\ &= \frac{2x^2y^2}{x^4y}\end{aligned}$$

Now, there several different ways you can reduce this answer to lowest terms, two of which are shown below.

You can factor numerator and denominator, then cancel common factors.

$$\begin{aligned}\frac{2x^2y^2}{x^4y} &= \frac{2\cdot x\cdot x\cdot y\cdot y}{x\cdot x\cdot x\cdot x\cdot y}\\ &= \frac{2\cdot \cancel{x}\cdot \cancel{x}\cdot \cancel{y}\cdot y}{\cancel{x}\cdot \cancel{x}\cdot x\cdot x\cdot \cancel{y}}\\ &= \frac{2y}{x^2}\end{aligned}$$

Or you can write the answer as a product, repeat the base and subtract exponents.

$$\begin{aligned}\frac{2x^2y^2}{x^4y} &= 2\cdot\frac{x^2}{x^4}\cdot\frac{y^2}{y^1}\\ &= 2x^{-2}y^1\\ &= \frac{2y}{x^2}\end{aligned}$$

In the last step, x^{-2} is the same as $1/x^2$, then we multiply numerators and denominators.

Note that the right-hand method is more efficient.

Exercise 7.3.2

Simplify: $\frac{3y}{x^3}\div\frac{y^2}{4x}$.

Answer

$\frac{12}{x^2y}$

Adding and Subtracting Rational Expressions

First, recall the rules for adding or subtracting fractions that have a “common” denominator.

Adding rational expressions

Given a/c and b/c, their sum is defined as:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$

That is, add the numerators and place the result over the common denominator.

The following examples each share a common denominator. We add the numerators, then place the result over the common denominator.

$$\frac{5}{7}+\frac{1}{7}=\frac{6}{7},\quad \frac{2}{x}+\frac{3}{x}=\frac{5}{x},\quad \text{and}\quad \frac{x}{y}+\frac{3y}{y}=\frac{x+3y}{y}$$

Example 7.3.3

Simplify: $\frac{3x}{xy}+\frac{2y}{xy}$.

Solution

Add the numerators, placing the result over the common denominator.

$$\frac{3x}{xy}+\frac{2y}{xy}=\frac{3x+2y}{xy}$$

Exercise 7.3.3

Simplify: $\frac{4x}{x^2y}+\frac{5y^2}{x^2y}$

Answer

$$\frac{4x+5y^2}{x^2y}$$

Subtracting rational expressions

Given a/c and b/c, their difference is defined as:

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

That is, subtract the numerators and place the result over the common denominator.

The following examples each share a common denominator. We subtract the numerators, then place the result over the common denominator.

$$\frac{7}{9}-\frac{2}{9}=\frac{2}{9},\quad \frac{5a}{b}-\frac{3a}{b}=\frac{2a}{b},\quad \text{and}\quad \frac{3x}{xy}-\frac{5y}{xy}=\frac{3x-5y}{xy}$$

As the next example shows, sometimes you may have to reduce your answer to lowest terms.

Example 7.3.4

Simplify: $\frac{5xy}{2z}-\frac{3xy}{2z}$.

Solution

Subtract the numerators, placing the result over the common denominator.

$$\begin{aligned}\frac{5xy}{2z}-\frac{3xy}{2z}&=\frac{5xy-3xy}{2z}\\&=\frac{2xy}{2z}\end{aligned}$$

To reduce to lowest terms, divide both numerator and denominator by 2.

$$\frac{xy}{z}$$

Exercise 7.3.4

Simplify: $\frac{8x}{3yz^2}-\frac{2x}{3yz^2}$.

Answer

$$\frac{2x}{yz^2}$$

The Least Common Denominator

When adding or subtracting, if the rational expressions do not share a common denominator, you must first make equivalent fractions with a common denominator.

Least common denominator

If the fractions a/b and c/d do not share a common denominator, then the least common denominator for b and d is defined as the smallest number (or expression) divisible by both b and d. In symbols, $\mathrm{LCD}(b,d)$ represents the least common denominator of b and d.

Example 7.3.5

Simplify: $\frac{x}{6}+\frac{2x}{9}$.

Solution

The smallest number divisible by both 6 and 9 is 18; i.e., $\mathrm{LCD}(6,9)=18$. We must first make equivalent fractions with a common denominator of 18.

$$\begin{aligned}\frac{x}{6}+\frac{2x}{9} &= \frac{x}{6}\cdot\frac{3}{3}+\frac{2x}{9}\cdot\frac{2}{2}\\ &= \frac{3x}{18}+\frac{4x}{18}\\ &= \frac{7x}{18}\end{aligned}$$

Exercise 7.3.5

Simplify: $\frac{3x}{8}+\frac{5x}{6}$.

Answer

$\frac{29x}{24}$

Example 7.3.6

Simplify: $\frac{y}{8x}-\frac{y}{12x}$.

Solution

The smallest expression divisible by both $8x$ and $12x$ is $24x$; i.e., $\mathrm{LCD}(8x,12x)=24x$. We must first make equivalent fractions with a common denominator of $24x$, then place the difference of the numerators over the common denominator.

$$\begin{aligned}\frac{y}{8x}-\frac{y}{12x} &= \frac{y}{8x}\cdot\frac{3}{3}-\frac{y}{12x}\cdot\frac{2}{2}\\ &= \frac{3y}{24x}-\frac{2y}{24x}\\ &= \frac{y}{24x}\end{aligned}$$

Exercise 7.3.6

Simplify: $\frac{x}{8y}-\frac{x}{10y}$.

Answer

$\frac{x}{40y}$

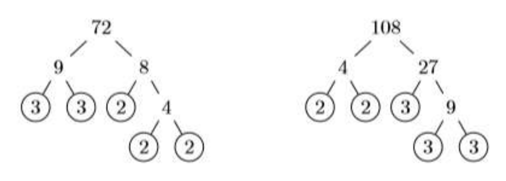
In Example $7.3.5$, it was not difficult to imagine the smallest number divisible by both 6 and 9. A similar statement might apply to Example $7.3.6$. This is not the case in all situations.

Example 7.3.7

Simplify: $\frac{5y}{72}-\frac{y}{108}$.

Solution

In this example, it is not easy to conjure up the smallest number divisible by both 72 and 108. As we shall see, prime factorization will come to the rescue.



Thus, $72=2^3\cdot 3^2$ and $108=2^2\cdot 3^3$.

Note: Procedure for finding the least common denominator (LCD)

To find the least common denominator for two or more fractions, proceed as follows:

1. Prime factor each denominator, putting your answers in exponential form.
2. To determine the LCD, write down each factor that appears in your prime factorizations to the highest power that it appears.

Following the procedure above, we list the prime factorization of each denominator in exponential form. The highest power of 2 that appears is 2^3. The highest power of 3 that appears is 3^3.

$$\begin{aligned} 72&=2^3\cdot 3^2 && \text{Prime factor 72.}\\ 108&=2^2\cdot 3^3 && \text{Prime factor 108.}\\ \mathrm{LCD}&=2^3\cdot 3^3 && \text{Highest power of 2 is } 2^3\text{. Highest power of 3 is } 3^3\text{.}\end{aligned}$$

Therefore, the LCD is $2^3\cdot 3^3=8\cdot 27$ or 216. Hence:

$$\begin{aligned}\frac{5y}{72}-\frac{y}{108}&=\frac{5y}{72}\cdot\frac{3}{3}-\frac{y}{108}\cdot\frac{2}{2} && \text{Make equivalent fractions.}\\ &=\frac{15y}{216}-\frac{2y}{216} && \text{Simplify.}\\ &=\frac{13y}{216} && \text{Subtract numerators.}\end{aligned}$$

Exercise 7.3.7

Simplify: $\frac{7x}{36}-\frac{3x}{40}$.

Answer

$$\frac{43x}{360}$$

Example 7.3.8

Simplify: $\frac{7}{15xy^2}-\frac{11}{20x^2}$

Solution

Prime factor each denominator, placing the results in exponential form.

$$\begin{aligned}15xy^2&=3\cdot 5\cdot x\cdot y^2\\ 20x^2&=2^2\cdot 5\cdot x^2\end{aligned}$$

To find the LCD, list each factor that appears to the highest power that it appears.

$$\mathrm{LCD}=2^2\cdot 3\cdot 5\cdot x^2\cdot y^2$$

Simplify.

$$\mathrm{LCD}=60x^2y^2$$

After making equivalent fractions, place the difference of the numerators over this common denominator.

$$\begin{aligned}\frac{7}{15xy^2}-\frac{11}{20x^2}&=\frac{7}{15xy^2}\cdot\frac{4x}{4x}-\frac{11}{20x^2}\cdot\frac{3y^2}{3y^2}\\&=\frac{28x}{60x^2y^2}-\frac{33y^2}{60x^2y^2}\\&=\frac{28x-33y^2}{60x^2y^2}\end{aligned}$$

Exercise 7.3.8

Simplify: $\frac{11}{18xy^2}+\frac{7x}{30xy}$

Answer

$\frac{55+21x^2}{90x^2y}$

Dividing a Polynomial by a Monomial

We know that multiplication is distributive with respect to addition; that is, $a(b+c)=ab+ac$. We use this property to perform multiplications such as:

$$x^2(2x^2-3x-8)=2x^4-3x^3-8x^2$$

However, it is also true that division is distributive with respect to addition.

Distributive property for division

If a, b, and c are any numbers, then:

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

For example, note that

$$\frac{4+6}{2}=\frac{4}{2}+\frac{6}{2}$$

This form of the distributive property can be used to divide a polynomial by a monomial.

Example 7.3.9

Divide x^2-2x-3 by x^2.

Solution

We use the distributive property, dividing each term by x^2.

$$\frac{x^2-2x-3}{x^2}=\frac{x^2}{x^2}-\frac{2x}{x^2}-\frac{3}{x^2}$$

Now we reduce each term of the last result to lowest terms, canceling common factors.

$$=1-\frac{2}{x}-\frac{3}{x^2}$$

Exercise 7.3.9

Divide $9x^3+8x^2-6x$ by $3x^2$.

Answer

$3x+\frac{8}{3}-\frac{2}{x}$

Example 7.3.10

Divide $2x^3 - 3x + 12$ by $6x^3$.

Solution

We use the distributive property, dividing each term by $6x^3$.

$$\frac{2x^3 - 3x + 12}{6x^3} = \frac{2x^3}{6x^3} - \frac{3x}{6x^3} + \frac{12}{6x^3}$$

Now we reduce each term of the last result to lowest terms, canceling common factors.

$$= \frac{1}{3} - \frac{1}{2x^2} + \frac{2}{x^3}$$

Exercise 7.3.10

Divide $-4x^2 + 6x - 9$ by $2x^4$.

Answer

$$-\frac{2}{x^2} + \frac{3}{x^3} - \frac{9}{x^4}$$

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7.4: Solving Rational Equations

In Section 3 of Chapter 2, we showed that the most efficient way to solve an equation containing fractions was to first clear the fractions by multiplying both sides of the equation by the least common denominator. For example, given the equation

$$\frac{1}{2}x+\frac{1}{3}=\frac{1}{4}$$

we would first clear the fractions by multiplying both sides by 12.

$$\begin{aligned} 12\left[\frac{1}{2}x+\frac{1}{3}\right] &= \left[\frac{1}{4}\right]12 \\ 6x+4 &= 3 \end{aligned}$$

This procedure works equally well when the the denominators contain a variable.

Example $7.4.1$

Solve for x: $1-\frac{2}{x}=\frac{3}{x^2}$

Solution

The common denominator is x^2. We begin by clearing fractions, multiplying both sides of the equation by x^2.

$$\begin{aligned} 1-\frac{2}{x} &= \frac{3}{x^2} && \text{Original equation.} \\ x^2\left[1-\frac{2}{x}\right] &= \left[\frac{3}{x^2}\right]x^2 && \text{Multiply both sides by } x^2. \end{aligned}$$

Now we use the distributive property.

$$x^2[1]-x^2\left[\frac{2}{x}\right]=\left[\frac{3}{x^2}\right]x^2 \quad \text{Distribute } x^2.$$

Now we cancel common factors and simplify.

$$x^2-2x=3 \quad \text{Cancel. Simplify.}$$

The resulting equation is nonlinear (x is raised to a power larger than 1). Make one side zero, then factor.

$$\begin{aligned} x^2-2x-3 &= 0 && \text{Nonlinear. Make one side zero.} \\ (x-3)(x+1) &= 0 && \text{Factor.} \end{aligned}$$

Use the zero product property to complete the solution. Either the first factor is zero or the second factor is zero.

$$\begin{aligned} x-3 &= 0 \\ x &= 3 \end{aligned}$$

or

$$\begin{aligned} x+1 &= 0 \\ x &= -1 \end{aligned}$$

Hence, the solutions are $x=-1$ and $x=3$.

Check. Substitute -1 for x, then 3 for x in the original equation and simplify.

$$\begin{aligned} 1-\frac{2}{x} &= \frac{3}{x^2} \\ 1-\frac{2}{(-1)^2} &= \frac{3}{(-1)^2} \\ 1+2 &= 3 \\ 3 &= 3 \end{aligned}$$

and

$$\begin{aligned} 1-\frac{2}{x} &= \frac{3}{x^2} \\ 1-\frac{2}{(\color{red}{3})} &= \frac{3}{(\color{red}{3})^2} \\ 1-\frac{2}{3} &= \frac{3}{9} \\ \frac{1}{3} &= \frac{1}{3} \end{aligned}$$

Note that both result in true statements, showing that both $x=-1$ and $x=3$ check in the original equation.

Exercise 7.4.1

Solve for x: $1-\frac{6}{x}=-\frac{8}{x^2}$

Answer

$2, 4$

Example 7.4.2

Solve for x: $6-\frac{22}{x^2}=\frac{29}{x}$

Solution

The common denominator is x^2.

$$\begin{aligned} 6-\frac{22}{x^2} &= \frac{29}{x} && \color{red}{\text{Original equation.}} \\ \color{red}{x^2}\left[6-\frac{22}{x^2}\right] &= \left[\frac{29}{x}\right]\color{red}{x^2} && \color{red}{\text{Multiply both sides by } x^2} \\ \color{red}{x^2}[6]-\color{red}{x^2}\left[\frac{22}{x^2}\right] &= \left[\frac{29}{x}\right]\color{red}{x^2} && \color{red}{\text{Distribute } x^2} \\ 6x^2-22 &= 29x && \color{red}{\text{Cancel and Simplify.}} \end{aligned}$$

This last equation is nonlinear. Make one side zero.

$$6x^2-29x-22=0$$

The integer pair 4 and -33 has product $ac=-132$ and sum $b=-29$. Break up the middle term into a sum using this pair, then factor by grouping.

$$\begin{aligned} 6x^2+4x-33x-22 &= 0 \\ 2x(3x+2)-11(3x+2) &= 0 \\ (2x-11)(3x+2) &= 0 \end{aligned}$$

Finally, use the zero product property to write:

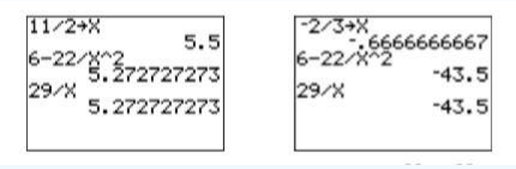
$$\begin{aligned} 2x-11 &= 0 \\ 2x &= 11 \\ x &= \frac{11}{2} \end{aligned}$$

or

$$\begin{aligned} 3x+2 &= 0 \\ 3x &= -2 \\ x &= -\frac{2}{3} \end{aligned}$$

Check: Let's check these solutions with our calculators. Enter $11/2$, push the STO► button, push the $\mathrm{X}, \mathrm{T}, \theta, \mathrm{n}$ button and the **ENTER** key (see the calculator screen on the left in Figure $7.4.1$). Next, enter the left-hand side of the equation as $6-22/X^2$ and press **ENTER**. Enter the right-hand side of the equation as $29/X$ and press **ENTER**. The results are the same (see the calculator screen on the left in Figure $7.4.1$). This verifies that $11/2$ is a solution of $6-22/x^2=29/x$.

The calculator screen on the right in Figure 7.4.1 shows a similar check of the solution $x = -2/3$.



```
11/2→X
              5.5
6-22/X^2
      5.272727273
29/X
      5.272727273
```

```
-2/3→X
     -.6666666667
6-22/X^2
            -43.5
29/X
            -43.5
```

Figure 7.4.1: Checking the solutions of $6-\frac{22}{x^2}=\frac{29}{x}$

Exercise 7.4.2

Solve for x: $\frac{7}{x^2}+8=-\frac{30}{x}$

Answer

$-1/4, -7/2$

Solving Rational Equations with the Graphing Calculator

Let's use the graphing calculator to solve an equation containing rational expressions.

Example 7.4.3

Consider the following equation:

$$2-\frac{9}{x}=\frac{5}{x^2}$$

Solve the equation algebraically, then solve the equation graphically using your graphing calculator. Compare your solutions.

Solution

Algebraic solution: First, an algebraic approach. Multiply both sides of the equation by the common denominator x^2.

$$\begin{aligned} 2-\frac{9}{x} &= \frac{5}{x^2} && \text{Original equation.} \\ x^2\left[2-\frac{9}{x}\right] &= \left[\frac{5}{x^2}\right]x^2 && \text{Multiply both sides by } x^2 \\ x^2[2]-x^2\left[\frac{9}{x}\right] &= \left[\frac{5}{x^2}\right]x^2 && \text{Distribute } x^2 \\ 2x^2-9x &= 5 && \text{Cancel and Simplify.} \end{aligned}$$

The last equation is nonlinear. Make one side zero.

$$2x^2-9x-5=0 \quad \text{Make one side zero.}$$

The integer pair -10 and 1 have product equaling $ac=-10$ and sum equaling $b=-9$. Break up the middle term using this pair, then factor by grouping.

$$\begin{aligned} 2x^2-10x+x-5 &= 0 && -10x+x=-9x \\ 2x(x-5)+1(x-5) &= 0 && \text{Factor by grouping.} \\ (2x+1)(x-5) &= 0 && \text{Factor out } x-5 \end{aligned}$$

Now use the zero product property to write:

$$\begin{aligned} 2x+1 &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

or

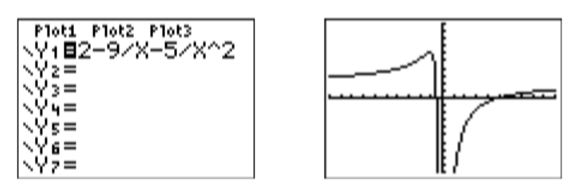
$$\begin{aligned} x-5 &= 0 \\ x &= 5 \end{aligned}$$

Hence, the solutions are $x=-1/2$ and $x=5$.

Graphical solution: We could load each side of the equation separately, then use the intersect utility to find where the graphs intersect. However, in this case, it's a bit easier to make one side of the equation zero, draw a single graph, then note where the graph crosses the x-axis.

$$\begin{aligned} 2-\frac{9}{x} &= \frac{5}{x^2} \quad \color{red}{\text{Original equation.}} \\ 2-\frac{9}{x}-\frac{5}{x^2} &= 0 \quad \color{red}{\text{Make one side zero.}} \end{aligned}$$

Load the left-hand side of the equation into $\mathrm{Y1}$ as $2-9/\mathrm{X}-5/\mathrm{X}\wedge 2$ (see the image on the left in Figure $7.4.2$), then select **6:ZStandard** from the **ZOOM** menu to produce the image at the right in Figure $7.4.2$.



```
Plot1 Plot2 Plot3
\Y1■2-9/X-5/X^2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

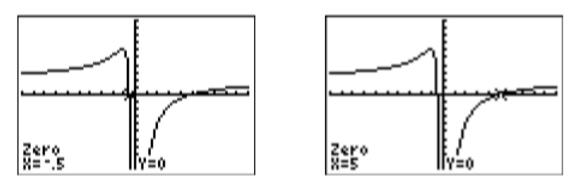
Figure $7.4.2$: Sketch the graph of $y=2-\frac{9}{x}-\frac{5}{x^2}$.

Next, the solutions of

$$2-\frac{9}{x}-\frac{5}{x^2}=0$$

are found by noting where the graph of $y=2-\frac{9}{x}-\frac{5}{x^2}$ cross the x-axis. Select **2:zero** from the **CALC** menu. Use the arrow keys to move the cursor to the left of the first x-intercept, then press **ENTER** to set the "Left bound." Next, move the cursor to the right of the first x-intercept, then press **ENTER** to set the "Right bound." Finally, leave the cursor where it is and press **ENTER** to set your "Guess." The calculator responds with the result shown in the figure on the left in Figure $7.4.3$.

Repeat the zero-finding procedure to capture the coordinates of the second x-intercept (see the image on the right in Figure $7.4.3$).



```
Zero
X=-.5        Y=0
```

```
Zero
X=5          Y=0
```

Figure $7.4.3$: Locating the zeros of $y=2-\frac{9}{x}-\frac{5}{x^2}$.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure $7.4.4$).
- Place your WINDOW parameters at the end of each axis (see Figure $7.4.4$).
- Label the graph with its equation (see Figure $7.4.4$).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $2-9/x-5/x^2=0$ (see Figure $7.4.4$).

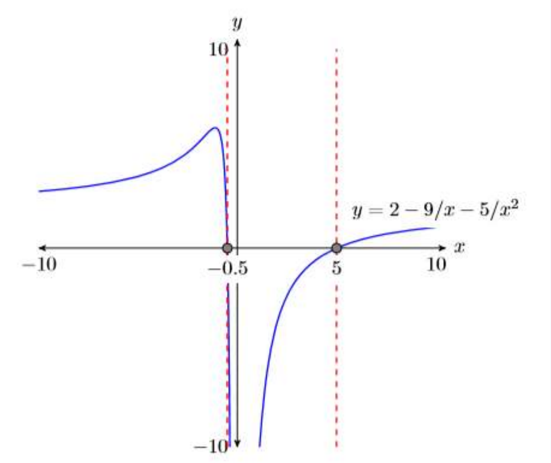


Figure $7.4.4$: Reporting your graphical solution on your homework.

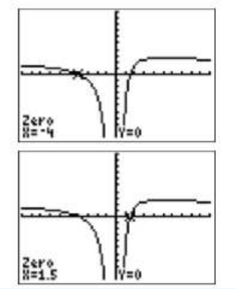
Thus, the calculator is reporting that the solutions of $2 - 9/x - 5/x^2 = 0$ are $x = -0.5$ and $x = 5$, which match the algebraic solutions $x = -1/2$ and $x = 5$.

Exercise $7.4.3$

Solve the equation $2 + \frac{5}{x} = \frac{12}{x^2}$ both algebraically and graphically, then compare your solutions.

Answer

$-4, 3/2$



Numerical Applications

Let’s apply what we’ve learned to an application.

Example $7.4.4$

The sum of a number and its reciprocal is $41/20$. Find the number.

Solution

In the solution, we address each step of the *Requirements for Word Problem Solutions*.

1. **Set up a variable dictionary:** Let x represent the unknown number.
2. **Set up an equation:** If the unknown number is x, then its reciprocal is $1/x$. Thus, the “sum of a number and its reciprocal is $41/20$” becomes:

$$x + \frac{1}{x} = \frac{41}{20}$$

3. **Solve the equation:** Clear the fractions by multiplying both sides by $20x$, the least common denominator.

$$\begin{aligned} x+\frac{1}{x} &= \frac{41}{20} && \text{Model equation.} \\ 20x\left[x+\frac{1}{x}\right] &= \left[\frac{41}{20}\right]20x && \text{Multiply both sides by } 20x \\ 20x[x]+20x\left[\frac{1}{x}\right] &= \left[\frac{41}{20}\right]20x && \text{Distribute } 20x. \\ 20x^2+20 &= 41x && \text{Cancel and simplify.} \end{aligned}$$

The equation is nonlinear. Make one side zero.

$$20x^2-41x+20=0 \quad \text{Make one side zero.}$$

The integer pair -16 and -25 has product $ac=400$ and sum $b=-41$. Break up the middle term in the last equation into a sum of like terms using this pair, then factor by grouping.

$$\begin{aligned} 20x^2-16x-25x+20 &= 0 && -16x-25x=-41x \\ 4x(5x-4)-5(5x-4) &= 0 && \text{Factor by grouping.} \\ (4x-5)(5x-4) &= 0 && \text{Factor out } 5x-4. \end{aligned}$$

We can now use the zero product property to write:

$$\begin{aligned} 4x-5 &= 0 \\ 4x &= 5 \\ x &= \frac{5}{4} \end{aligned}$$

or

$$\begin{aligned} 5x-4 &= 0 \\ 5x &= 4 \\ x &= \frac{4}{5} \end{aligned}$$

4. **Answer the question:** There are two possible numbers, $5/4$ and $4/5$.
5. **Look back:** The sum of the unknown number and its reciprocal is supposed to equal $41/20$. The answer $5/4$ has reciprocal $4/5$. Their sum is:

$$\begin{aligned} \frac{5}{4}+\frac{4}{5} &= \frac{16}{20}+\frac{25}{20} \\ &= \frac{41}{20} \end{aligned}$$

Thus, $5/4$ is a valid solution. The second answer $4/5$ has reciprocal $5/4$, so it is clear that their sum is also $41/20$. Hence, $4/5$ is also a valid solution.

Exercise 7.4.4

The sum of a number and its reciprocal is $53/14$. Find the number.

Answer

$2/7, 7/2$

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7.5: Direct and Inverse Variation

We start with the definition of the phrase “is proportional to.”

Proportional

We say that y is proportional to x if and only if

$$y = kx$$

where k is a constant called the constant of proportionality. The phrase “y varies directly as x” is an equivalent way of saying “ y is proportional to x.”

Here are a few examples that translate the phrase “is proportional to.”

- Given that d is proportional to t, we write $d = kt$, where k is a constant.
- Given that y is proportional to the cube of x, we write $y = kx^3$, where k is a constant.
- Given that s is proportional to the square of t, we write $s = kt^2$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

Example 7.5.1

Given that y is proportional to x and the fact that $y = 12$ when $x = 5$, determine the constant of proportionality, then determine the value of y when $x = 10$.

Solution

Given the fact the y is proportional to x, we know immediately that

$$y = kx$$

where k is the proportionality constant. Because we are given that $y = 12$ when $x = 5$, we can substitute 12 for y and 5 for x to determine k.

$$\begin{aligned} y &= kx && \text{y is proportional to x} \\ 12 &= k(5) && \text{Substitute 12 for y, 5 for x} \\ \frac{12}{5} &= k && \text{Divide both sides by 5} \end{aligned}$$

Next, substitute the constant of proportionality $12/5$ for k in $y = kx$, then substitute 10 for x to determine y when $x = 10$.

$$\begin{aligned} y &= \frac{12}{5}x && \text{Substitute 12/5 for k} \\ y &= \frac{12}{5}(10) && \text{Substitute 10 for x} \\ y &= 24 && \text{Cancel and simplify.} \end{aligned}$$

Exercise 7.5.1

Given that y is proportional to x and that $y = 21$ when $x = 9$, determine the value of y when $x = 27$.

Answer

63

Example 7.5.2

A ball is dropped from a balloon floating above the surface of the earth. The distance s the ball falls is proportional to the square of the time t that has passed since the ball’s release. If the ball falls 144 feet during the first 3 seconds, how far does the ball fall in 9 seconds?

Solution

Given the fact the s is proportional to the square of t, we know immediately that

$$s = kt^2$$

where k is the proportionality constant. Because we are given that the ball falls 144 feet during the first 3 seconds, we can substitute 144 for s and 3 for t to determine the constant of proportionality.

$$\begin{aligned} s &= kt^2 && \text{s is proportional to the square of t} \\ 144 &= k(3)^2 && \text{Substitute 144 for s, 3 for t} \\ 144 &= 9k && \text{Simplify: $3^2 = 9$} \\ 16 &= k && \text{Divide both sides by 9} \end{aligned}$$

Next, substitute the constant of proportionality 16 for k in $s = kt^2$, and then substitute 9 for t to determine the distance fallen when $t = 9$ seconds.

$$\begin{aligned} s &= 16t^2 && \text{Substitute 16 for k} \\ s &= 16(9)^2 && \text{Substitute 9 for t} \\ s &= 1296 && \text{Simplify} \end{aligned}$$

Thus, the ball falls $1,296$ feet during the first 9 seconds.

Exercise 7.5.2

A ball is dropped from the edge of a cliff on a certain planet. The distance s the ball falls is proportional to the square of the time t that has passed since the ball's release. If the ball falls 50 feet during the first 5 seconds, how far does the ball fall in 8 seconds?

Answer

128 feet

Example 7.5.3

Tony and Paul are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance y that the spring stretches is proportional to the weight hung on the spring (Hooke's Law). If a 0.5 pound weight stretches the spring 3 inches, how far will a 0.75 pound weight stretch the spring?

Solution

Let W represent the weight hung on the spring. Let y represent the distance the spring stretches. We're told that the distance y the spring stretches is proportional to the amount of weight W hung on the spring. Hence, we can write:

$$y = kW \qquad \text{y is proportional to W}$$

Substitute 3 for y, 0.5 for W, then solve fork.

$$\begin{aligned} 3 &= k(0.5) && \text{Substitute 3 for y, 0.5 for W} \\ \frac{3}{0.5} &= k && \text{Divide both sides by 0.5} \\ k &= 6 && \text{Simplify.} \end{aligned}$$

Substitute 6 for k in $y = kW$ to produce:

$$y = 6W \qquad \text{Substitute 6 for k in $y = kW$}$$

To determine the distance the spring will stretch when 0.75 pounds are hung on the spring, substitute 0.75 for W.

$$\begin{aligned} y &= 6(0.75) && \text{Substitute 0.75 for W} \\ y &= 4.5 && \text{Simplify.} \end{aligned}$$

Thus, the spring will stretch 4.5 inches.

Exercise 7.5.3

If a 0.75 pound weight stretches a spring 5 inches, how far will a 1.2 pound weight stretch the spring?

Answer

8 inches

Inversely Proportional

In Examples $7.5.1$, $7.5.2$, and $7.5.3$, where one quantity was proportional to a second quantity, you may have noticed that when one quantity increased, the second quantity also increased. Vice-versa, when one quantity decreased, the second quantity also decreased.

However, not all real-world situations follow this pattern. There are times when as one quantity increases, the related quantity decreases. For example, consider the situation where you increase the number of workers on a job and note that the time to finish the job decreases. This is an example of a quantity being inversely proportional to a second quantity.

Inversely proportional

We say the y is inversely proportional to x if and only if

$$y=\frac{k}{x}$$

where k is a constant called the constant of proportionality. The phrase "y varies inversely as x" is an equivalent way of saying "y in inversely proportional to x."

Here are a few examples that translate the phrase "is inversely proportional to."

- Given that d is inversely proportional to t, we write $d=k/t$, where k is a constant.
- Given that y is inversely proportional to the cube of x, we write $y=k/x^3$, where k is a constant.
- Given that s is inversely proportional to the square of t, we write $s=k/t^2$, where k is a constant.

We are not restricted to always using the letter k for our constant of proportionality.

Example $7.5.4$

Given that y is inversely proportional to x and the fact that $y=4$ when $x=2$, determine the constant of proportionality, then determine the value of y when $x=4$.

Solution

Given the fact the y is inversely proportional to x, we know immediately that

$$y=\frac{k}{x}$$

where k is the proportionality constant. Because we are given that $y=4$ when $x=2$, we can substitute 4 for y and 2 for x to determine k.

$$\begin{aligned} y&=\frac{k}{x} && \text{y is inversely proportional to x.}\\ 4&=\frac{k}{2} && \text{Substitute 4 for y, 2 for x.}\\ 8&=k && \text{Multiply both sides by 2.}\end{aligned}$$

Substitute 8 for k in $y=k/x$, then substitute 4 for x to determine y when $x=4$.

$$\begin{aligned} y&=\frac{8}{x} && \text{Substitute 8 for k.}\\ y&=\frac{8}{4} && \text{Substitute 4 for x.}\\ y&=2 && \text{Reduce.}\end{aligned}$$

Note that as x increased from 2 to 4, y decreased from 4 to 2.

Exercise 7.5.4

Given that y is inversely proportional to x and that $y = 5$ when $x = 8$, determine the value of y when $x = 10$.

Answer

4

Example 7.5.5

The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 5 feet from the light source is 3 foot-candles, what is the intensity of the light 15 feet from the light source?

Solution

Given the fact that the intensity I of the light is inversely proportional to the square of the distance d from the light source, we know immediately that

$$I = \frac{k}{d^2}$$

where k is the proportionality constant. Because we are given that the intensity is $I = 3$ foot-candles at $d = 5$ feet from the light source, we can substitute 3 for I and 5 for d to determine k.

$$\begin{aligned} I &= \frac{k}{d^2} && \color{red}{I \text{ is inversely proportional to } d^2.} \\ 3 &= \frac{k}{5^2} && \color{red}{\text{Substitute } 3 \text{ for } I, 5 \text{ for } d.} \\ 3 &= \frac{k}{25} && \color{red}{\text{Simplify.}} \\ 75 &= k && \color{red}{\text{Multiply both sides by } 25.} \end{aligned}$$

Substitute 75 for k in $I = k/d^2$, then substitute 15 for d to determine I when $d = 15$.

$$\begin{aligned} I &= \frac{75}{d^2} && \color{red}{\text{Substitute } 75 \text{ for } k.} \\ I &= \frac{75}{15^2} && \color{red}{\text{Substitute } 15 \text{ for } d.} \\ I &= \frac{75}{225} && \color{red}{\text{Simplify.}} \\ I &= \frac{1}{3} && \color{red}{\text{Reduce.}} \end{aligned}$$

Thus, the intensity of the light 15 feet from the light source is $1/3$ foot-candle.

Exercise 7.5.5

If the light intensity 4 feet from a light source is 2 foot-candles, what is the intensity of the light 8 feet from the light source?

Answer

$1/2$ foot-candle

Example 7.5.6

Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 10 people sign up, the price per person is $350. What will be the price per person if 50 people sign up?

Solution

Let p represent the price per person and let N be the number of people who sign up for the camping experience. Because we are told that the price per person is inversely proportional to the number of people who sign up for the camping experience, we can write:

$$p=\frac{k}{N}$$

where k is the proportionality constant. Because we are given that the price per person is $\$350$ when 10 people sign up, we can substitute 350 for p and 10 for N to determine k.

$$\begin{aligned} p &= \frac{k}{N} \quad \color{red}{p \text{ is inversely proportional to } N.} \\ 350 &= \frac{k}{10} \quad \color{red}{\text{Substitute } 350 \text{ for } p, 10 \text{ for } N.} \\ 3500 &= k \quad \color{red}{\text{Multiply both sides by } 10.} \end{aligned}$$

Substitute 3500 for k in $p=k/N$, then substitute 50 for N to determine p when $N=50$.

$$\begin{aligned} p &= \frac{3500}{N} \quad \color{red}{\text{Substitute } 3500 \text{ for } k.} \\ p &= \frac{3500}{50} \quad \color{red}{\text{Substitute } 50 \text{ for } N.} \\ p &= 70 \quad \color{red}{\text{Simplify.}} \end{aligned}$$

Thus, the price per person is $\$70$ if 50 people sign up for the camping experience.

Exercise 7.5.6

Suppose that the price per person for a tour is inversely proportional to the number of people who sign up for the tour. If 8 people sign up, the price per person is $\$70$. What will be the price per person if 20 people sign up?

Answer

$28

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7.E: Rational Expressions(Exercises)

7.1: Negative Exponents

In Exercises 1-8, simplify the given expression.

1) $\left(\frac{1}{7}\right)^{-1}$

Answer

7

2) $\left(-\frac{3}{5}\right)^{-1}$

3) $\left(-\frac{8}{9}\right)^{-1}$

Answer

$-\frac{9}{8}$

4) $\left(-\frac{3}{2}\right)^{-1}$

5) $(18)^{-1}$

Answer

$\frac{1}{18}$

6) $(-11)^{-1}$

7) $(16)^{-1}$

Answer

$\frac{1}{16}$

8) $(7)^{-1}$

In Exercises 9-16, simplify the given expression.

9) $a^{-9}a^{3}$

Answer

a^{-6}

10) $x^{-5}x^{-5}$

11) $b^{-9}b^{8}$

Answer

b^{-1}

12) $v^{-7}v^{-2}$

13) $2^{9}\cdot 2^{-4}$

Answer

2^{5}

14) $2^{2}\cdot 2^{-7}$

15) $9^{-6} \cdot 9^{-5}$

Answer

9^{-11}

16) $9^{7} \cdot 9^{-5}$

In Exercises 17-24, simplify the given expression.

17) $\frac{2^{6}}{2^{-8}}$

Answer

2^{14}

18) $\frac{6^{8}}{6^{-1}}$

19) $\frac{z^{-1}}{z^{9}}$

Answer

z^{-10}

20) $\frac{w^{-4}}{w^{3}}$

21) $\frac{w^{-9}}{w^{7}}$

Answer

w^{-16}

22) $\frac{r^{5}}{r^{-1}}$

23) $\frac{7^{-3}}{7^{-1}}$

Answer

7^{-2}

24) $\frac{6^{-8}}{6^{6}}$

In Exercises 25-32, simplify the given expression.

25) $(t^{-1})^{4}$

Answer

t^{-4}

26) $(a^{8})^{-7}$

27) $(6^{-6})^{7}$

Answer

6^{-42}

28) $(2^{-7})^{-7}$

29) $(z^{-9})^{-9}$

Answer

z^{81}

30) $\left(c^6\right)^{-2}$

31) $\left(3^{-2}\right)^3$

Answer

3^{-6}

32) $\left(8^{-1}\right)^6$

In Exercises 33-40, simplify the given expression.

33) 4^{-3}

Answer

$\frac{1}{64}$

34) 5^{-2}

35) 2^{-4}

Answer

$\frac{1}{16}$

36) $(-3)^{-4}$

37) $\left(\frac{1}{2}\right)^{-5}$

Answer

32

38) $\left(\frac{1}{3}\right)^{-3}$

39) $\left(-\frac{1}{2}\right)^{-5}$

Answer

-32

40) $\left(\frac{1}{2}\right)^{-4}$

In Exercises 41-56, simplify the given expression.

41) $\left(4u^{-6}v^{-9}\right)\left(5u^8v^{-8}\right)$

Answer

$20u^2v^{-17}$

42) $\left(6a^{-9}c^{-6}\right)\left(-8a^8c^5\right)$

43) $\left(6x^{-6}y^{-5}\right)\left(-4x^4y^{-2}\right)$

Answer

$-24x^{-2}y^{-7}$

44) $\left(5v^{-3}w^{-8}\right)\left(8v^{-9}w^5\right)$

45) $\dfrac{-6x^7z^9}{4x^{-9}z^{-2}}$

Answer

$-\dfrac{3}{2}x^{16}z^{11}$

46) $\dfrac{2u^{-2}v^6}{6u^2v^{-1}}$

47) $\dfrac{-6a^9c^6}{-4a^{-5}c^{-7}}$

Answer

$\dfrac{3}{2}a^{14}c^{13}$

48) $\dfrac{-4u^{-4}w^4}{8u^{-8}w^{-7}}$

49) $(2v^{-2}w^4)^{-5}$

Answer

$\dfrac{1}{32}v^{10}w^{-20}$

50) $(3s^{-6}t^5)^{-4}$

51) $(3x^{-1}y^7)^4$

Answer

$81x^{-4}y^{28}$

52) $(-4b^{-8}c^{-4})^3$

53) $(2x^6z^{-7})^5$

Answer

$32x^{30}z^{-35}$

54) $(-4v^4w^{-9})^3$

55) $(2a^{-4}c^8)^{-4}$

Answer

$\dfrac{1}{16}a^{16}c^{-32}$

56) $(11b^9c^{-1})^{-2}$

In Exercises 57-76, clear all negative exponents from the given expression.

57) $\dfrac{x^5y^{-2}}{z^3}$

Answer

$\dfrac{x^5}{y^2z^3}$

58) $\dfrac{x^4y^{-9}}{z^7}$

59) $\dfrac{r^9s^{-2}}{t^3}$

Answer

$$\frac{r^9}{s^2t^3}$$

60) $\frac{u^5v^{-3}}{w^2}$

61) $\frac{x^3}{y^{-8}z^5}$

Answer

$$\frac{x^3y^8}{z^5}$$

62) $\frac{x^9}{y^{-4}z^3}$

63) $\frac{u^9}{v^{-4}w^7}$

Answer

$$\frac{u^9v^4}{w^7}$$

64) $\frac{a^7}{b^{-8}c^6}$

65) $\left(7x^{-1}\right)\left(-7x^{-1}\right)$

Answer

$$\frac{-49}{x^2}$$

66) $\left(3a^{-8}\right)\left(-7a^{-7}\right)$

67) $\left(8a^{-8}\right)\left(7a^{-7}\right)$

Answer

$$\frac{56}{a^{15}}$$

68) $\left(-7u^3\right)\left(-8u^{-6}\right)$

69) $\frac{4x^{-9}}{8x^3}$

Answer

$$\frac{1}{2x^{12}}$$

70) $\frac{2t^{-8}}{-6t^9}$

71) $\frac{6c^2}{-4c^7}$

Answer

$$-\frac{3}{2c^5}$$

72) $\frac{6v^{-9}}{-8v^{-4}}$

73) $\left(-3s^9\right)^{-4}$

Answer

$$\frac{1}{81s^{36}}$$

74) $\left(-3 s^{8}\right)^{-4}$

75) $\left(2 y^{4}\right)^{-5}$

Answer

$$\frac{1}{32 y^{20}}$$

76) $\left(2 w^{4}\right)^{-5}$

7.2: Scientific Notation

In Exercises 1-8, write each of the following in decimal format.

1) 10^{-4}

Answer

0.0001

2) 10^{-13}

3) 10^{-8}

Answer

0.00000001

4) 10^{-9}

5) 10^{8}

Answer

$100,000,000$

6) 10^{14}

7) 10^{7}

Answer

$10,000,000$

8) 10^{9}

In Exercises 9-16, write each of the following in decimal format.

9) 6506399.9×10^{-4}

Answer

650.63999

10) 19548.4×10^{-2}

11) $3959.430928 \times 10^{2}$

Answer

395943.0928

12) 976.841866×10^{2}

13) 440906.28×10^{-4}

Answer

44.090628

14) 9147437.4×10^{-4}

15) 849.855115×10^{4}

Answer

8498551.15

16) 492.4414×10^{3}

In Exercises 17-24, convert each of the given numbers into scientific notation.

17) 390000

Answer

3.9×10^{5}

18) 0.0004902

19) 0.202

Answer

2.02×10^{-1}

20) 3231

21) 0.81

Answer

8.1×10^{-1}

22) 83400

23) 0.0007264

Answer

7.264×10^{-4}

24) 0.00395

In Exercises 25-32, convert each of the given expressions into scientific notation.

25) 0.04264×10^{-4}

Answer

4.264×10^{-6}

26) 0.0019×10^{-1}

27) 130000×10^{3}

Answer

1.3×10^{8}

28) 738×10^{-1}

29) 30.04×10^{5}

Answer

3.004×10^{6}

30) 76000×10^{-1}

31) 0.011×10^{1}

Answer

1.1×10^{-1}

32) 496000×10^{-3}

In Exercises 33-38, each of the following numbers are examples of numbers reported on the graphing calculator in scientific notation. Express each in plain decimal notation.

33) $1.134\mathrm{E}-1$

Answer

0.1134

34) $1.370\mathrm{E}-4$

35) $1.556\mathrm{E}-2$

Answer

0.01556

36) $1.802\mathrm{E}4$

37) $1.748\mathrm{E}-4$

Answer

0.0001748

38) $1.402\mathrm{E}0$

In Exercises 39-42, first, use the technique of Example 7.2.11 to approximate the given product without the use of a calculator. Next, use the MODE button to set you calculator in SCI and FLOAT mode, then enter the given product using scientific notation. When reporting your answer, report all digits shown in your calculator view screen.

39) $\left(2.5 \times 10^{-1}\right)\left(1.6 \times 10^{-7}\right)$

Answer

4×10^{-8}

40) $\left(2.91 \times 10^{-1}\right)\left(2.81 \times 10^{-4}\right)$

41) $\left(1.4 \times 10^{7}\right)\left(1.8 \times 10^{-4}\right)$

Answer

2.52×10^{3}

42) $\left(7.48 \times 10^{7}\right)\left(1.19 \times 10^{6}\right)$

In Exercises 43-46, first, use the technique of Example 7.2.12 to approximate the given quotient without the use of a calculator. Next, push the MODE button, then highlight SCI mode and press ENTER. Move your cursor to the same row containing the FLOAT command, then highlight the number 2 and press ENTER. This will round your answers to two decimal places. Press 2nd MODE to quit the MODE menu. With these settings, enter the given expression using scientific notation. When entering your answer, report all digits shown in the viewing window.

43) $\dfrac{3.2 \times 10^{-5}}{2.5 \times 10^{-7}}$

Answer

1.28×10^2

44) $\dfrac{6.47 \times 10^{-5}}{1.79 \times 10^8}$

45) $\dfrac{5.9 \times 10^3}{2.3 \times 10^5}$

Answer

2.57×10^{-2}

46) $\dfrac{8.81 \times 10^{-9}}{3.06 \times 10^{-1}}$

47) Overall the combined weight of biological material – animals, plants, insects, crops, bacteria, and so on – has been estimated to be at about 75 billion tons or 6.8×10^{13} kg (https://en.Wikipedia.org/wiki/Nature). If the Earth has mass of 5.9736×10^{24} kg, what is the percent of the Earth's mass that is made up of biomass?

Answer

1.14×10^{-11}

48) The Guinness World Record for the longest handmade noodle was set on March 20, 2011. The $1,704$-meter-long stretch of noodle was displayed during a noodle-making activity at a square in Southwest China's Yunnan province. Meigan estimates that the average width of the noodle (it's diameter) to be the same as her index finger or 1.5 cm. Using the volume formula for a cylinder ($V = \pi r^2 h$) estimate the volume of the noodle in cubic centimeters

49) Assume there are 1.43×10^6 miles of paved road in the United States. If you could travel at an average of 65 miles per hour nonstop, how many days would it take you to travel over all of the paved roads in the USA? How many years?

Answer

916.7 days, 2.5 yr

50) The population of the USA in mid-2011 was estimated to be 3.12×10^8 people and the world population at that time to be about 7.012×10^9 people. What percentage of the world population live in the USA?

7.3: Simplifying Rational Expressions

In Exercises 1-8, simplify each of the given experssions.

1) $\dfrac{12}{s^2} \cdot \dfrac{s^5}{9}$

Answer

$\dfrac{4s^3}{3}$

2) $\dfrac{6}{x^4} \cdot \dfrac{x^2}{10}$

3) $\dfrac{12}{v^3} \cdot \dfrac{v^4}{10}$

Answer

$\dfrac{6v}{5}$

4) $\dfrac{10}{t^4} \cdot \dfrac{t^5}{12}$

5) $\dfrac{s^5}{t^4} \div \dfrac{9s^2}{t^2}$

Answer

$$\frac{s^3}{9t^2}$$

6) $\frac{s^2}{t^2} \div \frac{6s^4}{t^4}$

7) $\frac{b^4}{c^4} \div \frac{9b^2}{c^2}$

Answer

$$\frac{b^2}{9c^2}$$

8) $\frac{b^5}{c^4} \div \frac{8b^2}{c^2}$

In Exercises 9-14, simplify each of the given expressions.

9) $-\frac{10s}{18} + \frac{19g}{18}$

Answer

$$\frac{s}{2}$$

10) $-\frac{14y}{2} + \frac{10y}{2}$

11) $\frac{5}{9c} - \frac{17}{9c}$

Answer

$$-\frac{4}{3c}$$

12) $\frac{19}{14r} - \frac{17}{14r}$

13) $-\frac{8x}{15yz} - \frac{16x}{15yz}$

Answer

$$-\frac{8x}{5yz}$$

14) $-\frac{17a}{20bc} - \frac{9a}{20bc}$

In Exercises 15-20, simplify each of the given expressions.

15) $\frac{9z}{10} + \frac{5z}{2}$

Answer

$$\frac{17z}{5}$$

16) $\frac{7u}{2} + \frac{11u}{6}$

17) $\frac{3}{10v} - \frac{4}{5v}$

Answer

$$-\frac{1}{2v}$$

18) $\frac{9}{10v} - \frac{7}{2v}$

19) $-\frac{8r}{5st} - \frac{9r}{10st}$

Answer

$-\frac{5r}{2st}$

20) $-\frac{7x}{6yz} - \frac{3x}{2yz}$

In Exercises 21-32, simplify each of the given expressions.

21) $\frac{11}{18rs^2} + \frac{5}{24r^2s}$

Answer

$\frac{44r+15s}{72r^2s^2}$

22) $\frac{5}{12uw^2} + \frac{13}{54u^2w}$

23) $\frac{5}{24rs^2} + \frac{17}{36r^2s}$

Answer

$\frac{15r+34s}{72r^2s^2}$

24) $\frac{13}{54vw^2} + \frac{19}{24v^2w}$

25) $\frac{7}{36y^3} + \frac{11}{48z^3}$

Answer

$\frac{28z^3+33y^3}{144y^3z^3}$

26) $\frac{19}{36x^3} + \frac{5}{48y^3}$

27) $\frac{5}{48v^3} + \frac{13}{36uv^3}$

Answer

$\frac{15w^3+52v^3}{144v^3w^3}$

28) $\frac{7}{72r^3} + \frac{17}{48s^3}$

29) $\frac{11}{50xy} - \frac{9}{40yz}$

Answer

$\frac{44z-45x}{200xyz}$

30) $\frac{9}{50rs} - \frac{13}{40st}$

31) $\frac{19}{50ab} - \frac{17}{40bc}$

Answer

$\frac{76c-85a}{200abc}$

32) $\frac{9}{50rs} - \frac{11}{40st}$

In Exercises 33-48, use the distributive property to divide each term in the numerator by the term in the denominator.

33) $\frac{6v+12}{3}$

Answer

$2v+4$

34) $\frac{28u+36}{4}$

35) $\frac{25u+45}{5}$

Answer

$5u+9$

36) $\frac{16x+4}{2}$

37) $\frac{2s-4}{s}$

Answer

$2-\frac{4}{g}$

38) $\frac{7r-8}{r}$

39) $\frac{3r-5}{r}$

Answer

$3-\frac{5}{r}$

40) $\frac{4u-2}{u}$

41) $\frac{3x^2-8x-9}{x^2}$

Answer

$3-\frac{8}{x}-\frac{9}{x^2}$

42) $\frac{4b^2-5b-8}{b^2}$

43) $\frac{2x^2-3x-6}{x^2}$

Answer

$2-\frac{3}{x}-\frac{6}{x^2}$

44) $\frac{6u^2-5u-2}{u^2}$

45) $\frac{12t^2+2t-16}{12t^2}$

Answer

$1+\frac{1}{6t}-\frac{4}{3t^2}$

46) $\dfrac{18b^2+9b-15}{18b^2}$

47) $\dfrac{4s^2+2s-10}{4s^2}$

Answer

$$1+\frac{1}{2s}-\frac{5}{2s^2}$$

48) $\dfrac{10w^2+12w-2}{10w^2}$

7.4: Solving Rational Equations

In Exercises 1-8, solve the equation.

1) $x=11+\dfrac{26}{x}$

Answer

$$-2, 13$$

2) $x=7+\dfrac{60}{x}$

3) $1-\dfrac{12}{x}=-\dfrac{27}{x^2}$

Answer

$$3, 9$$

4) $1+\dfrac{6}{x}=\dfrac{7}{x^2}$

5) $1-\dfrac{10}{x}=\dfrac{11}{x^2}$

Answer

$$11, -1$$

6) $1-\dfrac{20}{x}=-\dfrac{96}{x^2}$

7) $x=7+\dfrac{44}{x}$

Answer

$$-4, 11$$

8) $x=2+\dfrac{99}{x}$

In Exercises 9-16, solve the equation.

9) $12x=97-\dfrac{8}{x}$

Answer

$$8, \frac{1}{12}$$

10) $7x=-19-\dfrac{10}{x}$

11) $20+\dfrac{19}{x}=-\dfrac{3}{x^2}$

Answer

$-\frac{3}{4}, -\frac{1}{5}$

12) $33-\frac{8}{x}=\frac{1}{x^2}$

13) $8x=19-\frac{11}{x}$

Answer

$\frac{11}{8}, 1$

14) $28x=25-\frac{3}{x}$

15) $40+\frac{6}{x}=\frac{1}{x^2}$

Answer

$-\frac{1}{4}, \frac{1}{10}$

16) $18+\frac{11}{x}=-\frac{1}{x^2}$

In Exercises 17-20, solve each equation algebraically, then use the calculator to check your solutions.

17) $36x=-13-\frac{1}{x}$

Answer

$-\frac{1}{9}, -\frac{1}{4}$

18) $9x=43+\frac{10}{x}$

19) $14x=9-\frac{1}{x}$

Answer

$\frac{1}{2}, \frac{1}{7}$

20) $3x=16-\frac{20}{x}$

In Exercises 21-24, solve the equation algebraically, then solve the equation using the graphing calculator using the technique shown in Example 7.4.3. Report your solution using the Calculator Submission Guidelines demonstrated in Example 7.4.3.

21) $1-\frac{1}{x}=\frac{12}{x^2}$

Answer

$-3, 4$

22) $1+\frac{11}{x}=-\frac{28}{x^2}$

23) $2x=3+\frac{44}{x}$

Answer

$-4, \frac{11}{2}$

24) $2x=9-\frac{4}{x}$

25) The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number.

Answer

$2, \frac{1}{2}$

26) The sum of a number and its reciprocal is $\frac{65}{8}$. Find the number.

27) The sum of a number and 8 times its reciprocal is $\frac{17}{3}$. Find all possible solutions.

Answer

$3, \frac{8}{3}$

28) The sum of a number and 4 times its reciprocal is $\frac{17}{2}$. Find all possible solutions.

7.5: Direct and Inverse Variation

1) Given that s is proportional to t and the fact that $s = 632$ when $t = 79$, determine the value of s when $t = 50$.

Answer

400

2) Given that s is proportional to t and the fact that $s = 264$ when $t = 66$, determine the value of s when $t = 60$.

3) Given that s is proportional to the cube of t and the fact that $s = 1588867$ when $t = 61$, determine the value of s when $t = 63$.

Answer

1750329

4) Given that d is proportional to the cube of t and the fact that $d = 318028$ when $t = 43$, determine the value of d when $t = 76$.

5) Given that q is proportional to the square of c and the fact that $q = 13448$ when $c = 82$, determine the value of q when $c = 29$.

Answer

1682

6) Given that q is proportional to the square of c and the fact that $q = 3125$ when $c = 25$, determine the value of q when $c = 87$.

7) Given that y is proportional to the square of x and the fact that $y = 14700$ when $x = 70$, determine the value of y when $x = 45$.

Answer

6075

8) Given that y is proportional to the square of x and the fact that $y = 2028$ when $x = 26$, determine the value of y when $x = 79$.

9) Given that F is proportional to the cube of x and the fact that $F = 214375$ when $x = 35$, determine the value of F when $x = 36$.

Answer

233280

10) Given that d is proportional to the cube of t and the fact that $d = 2465195$ when $t = 79$, determine the value of d when $t = 45$.

11) Given that d is proportional to t and the fact that $d = 496$ when $t = 62$, determine the value of d when $t = 60$.

Answer

480

12) Given that d is proportional to t and the fact that $d = 405$ when $t = 45$, determine the value of d when $t = 65$.

13) Given that h is inversely proportional to x and the fact that $h = 16$ when $x = 29$, determine the value of h when $x = 20$.

Answer

$$\frac{116}{5}$$

14) Given that y is inversely proportional to x and the fact that $y = 23$ when $x = 15$, determine the value of y when $x = 10$.

15) Given that q is inversely proportional to the square of c and the fact that $q = 11$ when $c = 9$, determine the value of q when $c = 3$.

Answer

99

16) Given that s is inversely proportional to the square of t and the fact that $s = 11$ when $t = 8$, determine the value of s when $t = 10$.

17) Given that F is inversely proportional to x and the fact that $F = 19$ when $x = 22$, determine the value of F when $x = 16$.

Answer

$$\frac{209}{8}$$

18) Given that d is inversely proportional to t and the fact that $d = 21$ when $t = 16$, determine the value of d when $t = 24$.

19) Given that y is inversely proportional to the square of x and the fact that $y = 14$ when $x = 4$, determine the value of y when $x = 10$.

Answer

$$\frac{56}{25}$$

20) Given that d is inversely proportional to the square of t and the fact that $d = 21$ when $t = 8$, determine the value of d when $t = 12$.

21) Given that d is inversely proportional to the cube of t and the fact that $d = 18$ when $t = 2$, determine the value of d when $t = 3$.

Answer

$$\frac{16}{3}$$

22) Given that q is inversely proportional to the cube of c and the fact that $q = 10$ when $c = 5$, determine the value of q when $c = 6$.

23) Given that q is inversely proportional to the cube of c and the fact that $q = 16$ when $c = 5$, determine the value of q when $c = 6$.

Answer

$$\frac{250}{27}$$

24) Given that q is inversely proportional to the cube of c and the fact that $q = 15$ when $c = 6$, determine the value of q when $c = 2$.

25) Joe and Mary are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance that the spring stretches is proportional to the weight hung on the spring. If a 2 pound weight stretches the spring 16 inches, how far will a 5 pound weight stretch the spring?

Answer

40 inches

26) Liz and Denzel are hanging weights on a spring in the physics lab. Each time a weight is hung, they measure the distance the spring stretches. They discover that the distance that the spring stretches is proportional to the weight hung on the spring. If a 5 pound weight stretches the spring 12.5 inches, how far will a 12 pound weight stretch the spring?

27) The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 4 feet from the light source is 20 foot-candles, what is the intensity of the light 18 feet from the light source?

Answer

1.0 foot-candles

28) The intensity I of light is inversely proportional to the square of the distance d from the light source. If the light intensity 5 feet from the light source is 10 foot-candles, what is the intensity of the light 10 feet from the light source

29) Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 18 people sign up, the price per person is $\$204$. What will be the price per person if 35 people sign up? Round your answer to the nearest dollar.

Answer

$\$105$

30) Suppose that the price per person for a camping experience is inversely proportional to the number of people who sign up for the experience. If 17 people sign up, the price per person is $\$213$. What will be the price per person if 27 people sign up? Round your answer to the nearest dollar.

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CHAPTER OVERVIEW

8: Quadratic Functions

8.1: Introduction to Radical Notation
8.2: Simplifying Radical Expressions
8.3: Completing the Square
8.4: The Quadratic Formula
8.E: Quadratic Functions (Exercises)

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8.1: Introduction to Radical Notation

We know how to square a number. For example:

- $5^2 = 25$
- $(-5)^2 = 25$

Taking the square root of a number is the opposite of squaring.

- The nonnegative square root of 25 is 5.
- The negative square root of 25 is -5.

Thus, when searching for a square root of a number, we are searching for number whose square is equal to our number.

Example $8.1.1$

Find the square roots of 81.

Solution

We are looking for a number whose square is 81.

- Because $9^2 = 81$, the nonnegative square root of 81 is 9.
- Because $(-9)^2 = 81$, the negative square root of 81 is -9.

Hence, 81 has two square roots, -9 and 9.

Exercise $8.1.1$

Find the square roots of 64.

Answer

8 and -8

Example $8.1.2$

Find the square roots of 0.

Solution

We are looking for a number whose square is 0.

- Because $0^2 = 0$, the nonnegative square root of 0 is 0.

No other number squared will equal zero. Hence, zero has exactly one square root, namely zero.

Exercise $8.1.2$

Find the square roots of 100.

Answer

10 and -10

Example $8.1.3$

Find the square roots of -36.

Solution

We are looking for a number whose square is -36. However, every time you square a real number, the result is never negative. Hence, -36 has no real square roots.[1]

Exercise 8.1.3

Find the square roots of -25.

Answer

no real square roots

The introductions in Examples 8.1.1, 8.1.2, and 8.1.3 lead to the following definition.

Defining the Square Roots of a Number

The solutions of $x^2 = a$ are called square roots of a.

Case: $a > 0$. The equation $x^2 = a$ has two real solutions, namely $x = \pm\sqrt{a}$.

- The notation$\sqrt{a}$ calls for the nonnegative square root.
- The notation $-\sqrt{a}$ calls for the negative square root.

Case: $a = 0$. The equation $x^2 = 0$ has exactly one solution, namely $x = 0$.

Case: $a < 0$. The equation $x^2 = a$ has no real solutions.

Example 8.1.4

Solve $x^2 = 9$ for x, then simplify your answers.

Solution

Because the right-hand side of $x^2 = 9$ is positive, the equation has two solutions.

$$\begin{aligned} x^2 &= 9 \quad \text{Original equation.} \\ x &= \pm\sqrt{9} \quad \text{Two answers: } -\sqrt{9} \text{ and } \sqrt{9} \end{aligned}$$

To simplify these answers, we need to understand the following facts:

- $\sqrt{9}$ calls for the nonnegative square root of 9. Because $(3)^2 = 9$, the nonnegative square root of 9 is 3. Hence, $\sqrt{9} = 3$.
- $-\sqrt{9}$ calls for the negative square root of 9. Because $(-3)^2 = 9$, the negative square root of 9 is -3. Hence, $-\sqrt{9} = -3$.

Thus, the solutions of $x^2 = 9$ are $x = \pm 3$, which is equivalent to saying "$x = -3$ or $x = 3$."

Exercise 8.1.4

Solve $x^2 = 16$ for x, then simplify your answers.

Answer

$4, -4$

Example 8.1.5

Solve $x^2 = 0$ for x, then simplify the answer.

Solution

There is only one number whose square equals 0, namely 0.

$$\begin{aligned} x^2 &= 0 \quad \text{Original equation.} \\ x &= 0 \quad \text{One answer: } (0)^2 = 0. \end{aligned}$$

Thus, the only solution of $x^2 = 0$ is $x = 0$. Consequently, the nonnegative square root of zero is zero. Hence, $\sqrt{0} = 0$.

Exercise 8.1.5

Solve $x^2 = 49$ for x, then simplify the answers.

Answer

$7, -7$

Example 8.1.6

Solve $x^2 = -4$ for x, then simplify the answer.

Solution

You cannot square a real number and get a negative result. Hence, $x^2 = -4$ has no real solutions. Therefore, $\sqrt{-4}$ is not a real number.

Exercise 8.1.6

Solve $x^2 = -9$ for x, then simplify the answers.

Answer

no real solutions

Example 8.1.7

Simplify each of the following:

a. $\sqrt{121}$
b. -$\sqrt{225}$
c. $\sqrt{-100}$
d. -$\sqrt{324}$

Solution

Remember, the notation $\sqrt{a}$ calls for the nonnegative square root of a, while the notation -$\sqrt{a}$ calls for the negative square root of a.

1. Because $11^2 = 121$, the nonnegative square root of 225 is -15. Thus:

$$-\sqrt{225} = -15$$

2. Because $(-15)^2 = 225$, the nonnegative square root of 121 is 11. Thus:

$$\sqrt{121} = 11$$

3. You cannot square a real number and get -100. Therefore, $\sqrt{-100}$ is not a real number.
4. Because $(-18)^2 = 324$, the nonnegative square root of 324 is -18. Thus:

$$-\sqrt{324} = -18$$

Exercise 8.1.7

Simplify: $-\sqrt{144}$

Answer

-12

Squaring “undoes” taking the square root.

Squaring square roots

If $a>0$, then both $-\sqrt{a}$ and $\sqrt{a}$ are solutions of $x^2=a$. Consequently, if we substitute each of them into the equation $x^2=a$, we get:

$$(-\sqrt{a})^2=a \text{ and } (\sqrt{a})^2=a$$

Example $8.1.8$

Simplify each of the following expressions:

a. $(\sqrt{5})^2$
b. $(-\sqrt{7})^2$
c. $(\sqrt{-11})^2$

Solution

We'll handle each case carefully.

a. Because $\sqrt{5}$ is a solution of $x^2=5$, if we square $\sqrt{5}$, we should get 5.

$$(\sqrt{5})^2=5$$

b. Because $-\sqrt{7}$ is a solution of $x^2=7$, if we square $-\sqrt{7}$, we should get 7.

$$(-\sqrt{7})^2=7$$

c. Because $x^2=-11$ has no real answers, $\sqrt{-11}$ is not a real number. Advanced courses such as college algebra or trigonometry will introduce the complex number system and show how to handle this expression.

Exercise $8.1.8$

Simplify: $(-\sqrt{21})^2$

Answer

21

Using the Graphing Calculator

Up to this point, the equation $x^2=a$ has involved perfect squares. For example, if we start with $x^2=25$, then the solutions are $x=\pm\sqrt{25}$. Because 25 is a perfect square, we can simplify further, arriving at $x=\pm 5$.

However, the right-hand side of $x^2=a$ does not have to be a perfect square. For example, the equation $x^2=7$ has two real solutions, $x=\pm\sqrt{7}$. Because 7 is not a perfect square, we cannot simplify further. In the next example, we'll use the graphing calculator to compare this algebraic solution with a graphical solution and hopefully provide some assurance that $-\sqrt{7}$ and $\sqrt{7}$ are perfectly valid solutions of $x^2=7$.

Example $8.1.9$

Use the graphing calculator to solve $x^2=7$. Then solve the equation algebraically and compare answers.

Solution

Enter each side of the equation $x^2=7$ in the **Y= menu** (see Figure $8.1.1$), then select **6:ZStandard** to produce the image in Figure $8.1.1$.

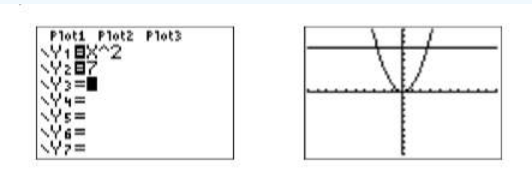
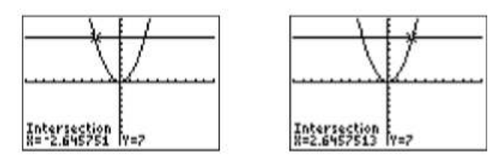


Figure $8.1.1$: Sketch each side of $x^2=7$.

Use the **5:intersect** utility on the **CALC** menu to find the points of intersection. Press **ENTER** in response to “First curve,” press **ENTER** in response to “Second curve,” then use the arrow keys to move the cursor closer to the point of intersection on the left than the one on the right. Press **ENTER** in response to “Guess.” This will produce the point of intersection shown in image on the left in Figure $8.1.2$. Repeat the procedure to find the point of intersection in the image on the right in Figure $8.1.2$.



```
Intersection
X=-2.645751  Y=7
```

```
Intersection
X=2.6457513  Y=7
```

Figure $8.1.2$: Finding points of intersection.

The approximate solutions of $x^2 = 7$ are $x \approx -2.645751$ and $x \approx 2.6457513$.

Reporting the solution on your homework: Duplicate the image in your calculator’s viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure $8.1.3$).
- Place your **WINDOW** parameters at the end of each axis (see Figure $8.1.3$).
- Label each graph with its equation (see Figure $8.1.3$).
- Drop dashed vertical lines through each point of intersection. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2 = 7$ (see Figure $8.1.3$).

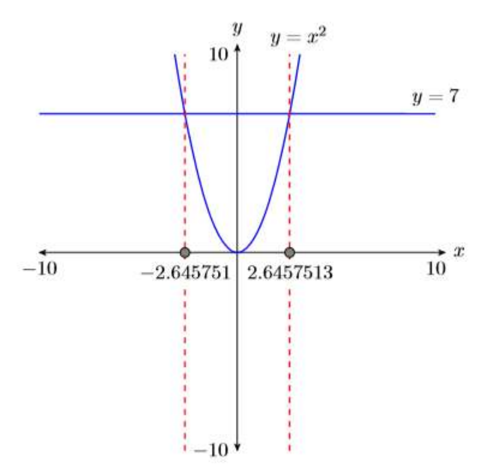
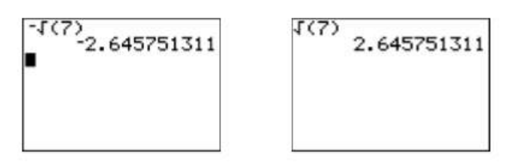


Figure $8.1.3$: Reporting your graphical solution on your homework.

Now we solve the equation algebraically.

$$\begin{aligned} x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

At this point, the question is: “Do these algebraic solutions match the graphical solutions in Figure $8.1.3$?” Let’s use our calculator to compare results. Locate the square root symbol $\sqrt{}$ on the calculator case above the x^2 key in the leftmost column on the calculator keyboard. Note that we will have to use the 2^{nd} key to access this operator. Enter $-\sqrt{(7)}$ and press **ENTER**. Then enter $\sqrt{(7)}$ and press **ENTER**. The results are shown in Figure $8.1.4$.



```
-√(7)
     -2.645751311
█
```

```
√(7)
      2.645751311
```

Figure $8.1.4$: Computing $-\sqrt{7}$ and $\sqrt{7}$.

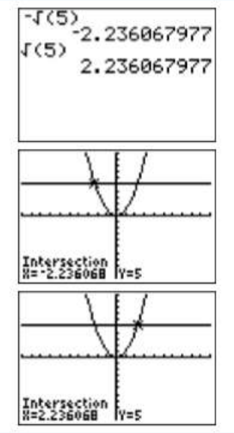
Thus, $-\sqrt{7} \approx -2.645751311$ and $\sqrt{7} \approx 2.645751311$. Note how these closely match the graphical approximations in Figure $8.1.3$.

Exercise 8.1.9

Solve the equation $x^2 = 5$ both algebraically and graphically, then compare your answers.

Answer

$-\sqrt{5}$, $\sqrt{5}$



```
-√(5)
         -2.236067977
√(5)
          2.236067977
```

```
Intersection
X=-2.236068  Y=5
```

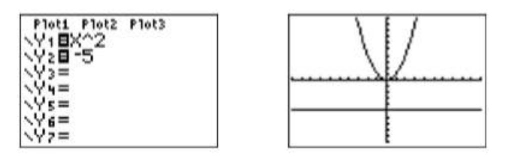
```
Intersection
X=2.236068   Y=5
```

Example 8.1.10

Use the graphing calculator to solve $x^2 = -5$. Then solve the equation algebraically and compare answers.

Solution

Enter each side of the equation $x^2 = -5$ in the **Y= menu** (see Figure 8.1.5), then select **6:ZStandard** to produce the image in Figure 8.1.5.



```
Plot1  Plot2  Plot3
\Y1 ■X^2
\Y2 ■-5
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =
```

Figure 8.1.5: Sketch each side of $x^2 = -5$.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure 8.1.6).
- Place your WINDOW parameters at the end of each axis (see Figure 8.1.6).
- Label each graph with its equation (see Figure 8.1.6).

Because there are no points of intersection, the graph in Figure 8.1.6 informs us that the equation $x^2 = -5$ has no real solutions. Now we solve the equation algebraically. $x^2 = -5$ However, you cannot square a real number and get a negative answer. Hence, the equation $x^2 = -5$ has no real solutions. This agrees completely with the graph in Figure 8.1.6.

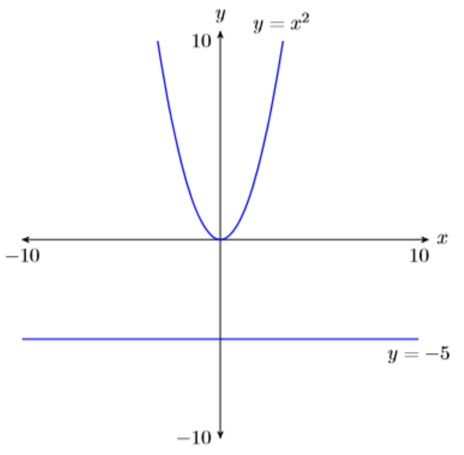


Figure 8.1.6: Reporting your graphical solution on your homework.

Approximating Square Roots

Table 8.1.1: List of Squares

n	n^2
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
11	121
12	144
13	169
14	196
15	225
16	256
17	289
18	324
19	361
20	400
21	441

n	n^2
22	484
23	529
24	576
25	625

The squares in the "List of Squares" shown in Table $8.1.1$ are called *perfect squares*. Each is the square of a whole number. But not all numbers are perfect squares. For example, in the case of $\sqrt{24}$, there is no whole number whose square is equal to 24. However, this does not prevent $\sqrt{24}$ from being a perfectly good number.

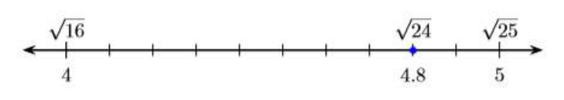
We can use the "List of Squares" to find decimal approximations when the radicand is not a perfect square.

Example $8.1.11$

Estimate $\sqrt{24}$ by guessing. Use a calculator to find a more accurate result and compare this result with your guess.

Solution

From the "List of Squares," note that 24 lies between 16 and 25, so $\sqrt{24}$ will lie between 4 and 5, with $\sqrt{24}$ much closer to 5 than it is to 4.



Let's guess

$$\sqrt{24} \approx 4.8$$

. As a check, let's square

$$4.8.(4.8)^2 = (4.8)(4.8) = 23.04$$

Not quite 24! Clearly, $\sqrt{24}$ must be a little bit bigger than 4.8.

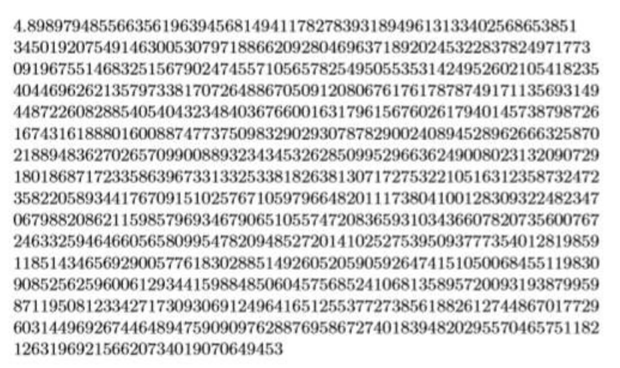
Let's use a scientific calculator to get a better approximation. From our calculator, using the square root button, we find

$$\sqrt{24} \approx 4.89897948557$$

.

Even though this is better than our estimate of 4.8, it is still only an approximation. Our calculator was only capable of providing 11 decimal places. However, the exact decimal representation of $\sqrt{24}$ is an infinite decimal that never terminates and never establishes a pattern of repetition.

Just for fun, here is a decimal approximation of $\sqrt{24}$ that is accurate to 1000 places, courtesy of www.wolframalpha.com



4.8989794855663561963945681494117827839318949613133402568653851
3450192075491463005307971886620928046963718920245322837824971773
091967551468325156790247455710565782549505553531424952602105418235
40446962621357973381707264886705091208067617617878749171135693149
44872260828854054043234840367660016317961567602617940145738798726
16743161888016008874773750983290293078782900240894528962666325870
21889483627026570990088932343453262850995296636249008023132090729
18018687172335863967331332533818263813071727532210516312358732472
35822058934417670915102576710597966482011173804100128309322482347
06798820862115985796934679065105574720836593103436607820735600767
24633259464660565809954782094852720141025275395093777354012819859
11851434656929005776183028851492605205905926474151050068455119830
90852562596006129344159884850604575685241068135895720093193879959
87119508123342717309306912496416512553772738561882612744867017729
60314496926744648947590909762887695867274018394820295570465751182
126319692156620734019070649453

If you were to multiply this number by itself (square the number), you would get a number that is extremely close to 24, but it would not be exactly 24. There would still be a little discrepancy.

Exercise 8.1.11

Estimate: $\sqrt{83}$

Answer

9.1

Reference

1 When we say that -36 has no real square roots, we mean there are no real numbers that are square roots of -36. The reason we emphasize the word real in this situation is the fact that -36 does have two square roots that are elements of the complex numbers, a set of numbers that are usually introduced in advanced courses such as college algebra or trigonometry.

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8.2: Simplifying Radical Expressions

Let’s begin by comparing two mathematical expressions.

$$\begin{aligned}\sqrt{9}\sqrt{16} &= 3 \cdot 4\\ &= 12\end{aligned}$$

and

$$\begin{aligned}\sqrt{9 \cdot 16} &= \sqrt{144}\\ &= 12\end{aligned}$$

Note that both $\sqrt{9}\sqrt{16}$ and $\sqrt{9 \cdot 16}$ equal 12. Hence, $\sqrt{9}\sqrt{16} = \sqrt{9 \cdot 16}$. Let’s look at another example.

$$\begin{aligned}\sqrt{4}\sqrt{9} &= 2 \cdot 3\\ &= 6\end{aligned}$$

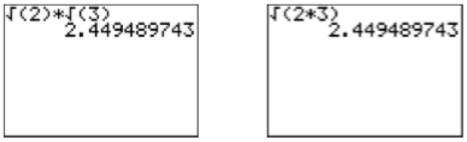
and

$$\begin{aligned}\sqrt{4 \cdot 9} &= \sqrt{36}\\ &= 6\end{aligned}$$

Note that both $\sqrt{4}\sqrt{9}$ and $\sqrt{4 \cdot 9}$ equal 6. Hence, $\sqrt{4}\sqrt{9} = \sqrt{4 \cdot 9}$. It appears that a pattern is forming, namely:

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

Let’s try an example on our calculator. First enter $\sqrt{2}\sqrt{3}$, then enter $\sqrt{2 \cdot 3}$ (see Figure $8.2.1$). Note that they produce the same result. Therefore, $\sqrt{2}\sqrt{3} = \sqrt{2 \cdot 3}$



```
√(2)*√(3)
             2.449489743
```

```
√(2*3)
             2.449489743
```

Figure $8.2.1$: Note that $\sqrt{2}\sqrt{3} = \sqrt{2 \cdot 3}$

The above discussion leads us to the following result.

Multiplication property of radicals

If $a \geq 0$ and $b \geq 0$, then:

$$\sqrt{a}\sqrt{b} = \sqrt{a \cdot b}$$

Example $8.2.1$

Simplify each of the following expressions as much as possible:

a. $\sqrt{3}\sqrt{11}$
b. $\sqrt{12}\sqrt{3}$
c. $\sqrt{2}\sqrt{13}$

Solution

In each case, use the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$. That is, multiply the two numbers under the square root sign, placing the product under a single square root.

a. $$\begin{aligned}\sqrt{3}\sqrt{11} &= \sqrt{3 \cdot 11}\\ &= \sqrt{33}\end{aligned}$$

b. $$\begin{aligned}\sqrt{12}\sqrt{3} &= \sqrt{12 \cdot 3}\\ &= \sqrt{36}\\ &= 6\end{aligned}$$

c. $\sqrt{2}\sqrt{13} = \sqrt{2 \cdot 13}$
$= \sqrt{26}$

Exercise 8.2.1

Simplify: $\sqrt{2}\sqrt{8}$

Answer

4

Simple Radical Form

We can also use the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ in reverse to factor out a perfect square. For example:

$$\begin{aligned} \sqrt{50} &= \sqrt{25}\sqrt{2} && \text{Factor out a perfect square.} \\ &= 5\sqrt{2} && \text{Simplify: } \sqrt{25} = 5 \end{aligned}$$

The expression $5\sqrt{2}$ is said to be in **simple radical form**. Like reducing a fraction to lowest terms, you should always look to factor out a perfect square when possible.

Simple Radical Form

If possible, always factor out a perfect square.

Example 8.2.2

Place $\sqrt{8}$ in simple radical form.

Solution

From $\sqrt{8}$, we can factor out a perfect square, in this case $\sqrt{4}$.

$$\begin{aligned} \sqrt{8} &= \sqrt{4}\sqrt{2} && \text{Factor out a perfect square.} \\ &= 2\sqrt{2} && \text{Simplify: } \sqrt{4} = 2 \end{aligned}$$

Exercise 8.2.2

Place $\sqrt{12}$ in simple radical form.

Answer

$2\sqrt{3}$

Sometimes, after factoring out a perfect square, you can still factor out another perfect square.

Example 8.2.3

Place $\sqrt{72}$ in simple radical form.

Solution

From $\sqrt{72}$, we can factor out a perfect square, in this case $\sqrt{9}$.

$$\begin{aligned} \sqrt{72} &= \sqrt{9}\sqrt{8} && \text{Factor out a perfect square.} \\ &= 3\sqrt{8} && \text{Simplify: } \sqrt{9} = 3 \end{aligned}$$

However, from $\sqrt{8}$ we can factor out another perfect square, in this case $\sqrt{4}$.

$$\begin{aligned} &= 3\sqrt{4}\sqrt{2} && \text{Factor out another perfect square.} \\ &= 3 \cdot 2 \cdot \sqrt{2} && \text{Simplify: } \sqrt{4} = 2 \\ &= 6\sqrt{2} && \text{Multiply: } 3 \cdot 2 = 6 \end{aligned}$$

Alternate solution

We can simplify the process by noting that we can factor out $\sqrt{36}$ from $\sqrt{72}$ to start the process.

$$\begin{aligned} \sqrt{72} &= \sqrt{36}\sqrt{2} && \color{red}{\text{Factor out a perfect square.}} \\ &= 6\sqrt{2} && \color{red}{\text{Simplify: } \sqrt{36} = 6} \end{aligned}$$

Although the second solution is more efficient, the first solution is still mathematically correct. The point to make here is that we must continue to factor out a perfect square whenever possible. Our answer is not in simple radical form until we can no longer factor out a perfect square.

Exercise $8.2.3$

Place $\sqrt{200}$ in simple radical form.

Answer

$10\sqrt{2}$

The Pythagorean Theorem

An angle that measures 90 degrees is called a *right angle*. If one of the angles of a triangle is a right angle, then the triangle is called a right triangle. It is traditional to mark the right angle with a little square (see Figure $8.2.2$).

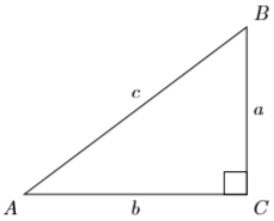


Figure $8.2.2$: Right triangle $\triangle ABC$ has a right angle at vertex C.

Right Triangle Terminology

- The longest side of the right triangle, the side directly opposite the right angle, is called the hypotenuse of the right triangle.
- The remaining two sides of the right triangle are called the legs of the right triangle.

Proof of the Pythagorean Theorem

Each side of the square in Figure $8.2.3$ has been divided into two segments, one of length a, the other of length b.

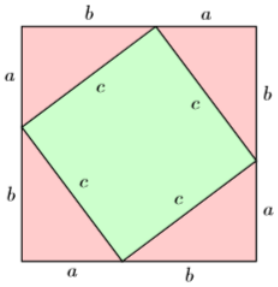


Figure $8.2.3$: Proving the Pythagorean Theorem.

We can find the total area of the square by squaring any one of the sides of the square.

$$\begin{aligned} A &= (a+b)^2 && \color{red}{\text{Square a side to find area.}} \\ A &= a^2 + 2ab + b^2 && \color{red}{\text{Squaring a binomial pattern.}} \end{aligned}$$

Thus, the total area of the square is $A = a^2 + 2ab + b^2$.

A second approach to finding the area of the square is to sum the areas of the geometric parts that make up the square. We have four congruent right triangles, shaded in light red, with base a and height b. The area of each of these triangles is found by taking one-half times the base times the height; i.e., the area of each triangles is $(1/2)ab$. In the interior, we have a smaller square with side c. Its area is found by squaring its side; i.e., the area of the smaller square is c^2.

The total area of the square is the sum of its parts, one smaller square and four congruent triangles. That is:

$$\begin{aligned} A &= c^2 + 4\left(\tfrac{1}{2}ab\right) && \text{Adding the area of the interior square and the area of four right triangles.} \\ A &= c^2 + 2ab && \text{Simplify: } 4((1/2)ab) = 2ab \end{aligned}$$

The two expressions, $a^2 + 2ab + b^2$ and $c^2 + 2ab$, both represent the total area of the large square. Hence, they must be equal to one another.

$$\begin{aligned} a^2 + 2ab + b^2 &= c^2 + 2ab && \text{Each side of this equation represents the area of the large square.} \\ a^2 + b^2 &= c^2 && \text{Subtract } 2ab \text{ from both sides.} \end{aligned}$$

The last equation, $a^2 + b^2 = c^2$, is called the Pythagorean Theorem.

The Pythagorean Theorem

If a and b are the legs of a right triangle and c is its hypotenuse, then:

$$a^2 + b^2 = c^2$$

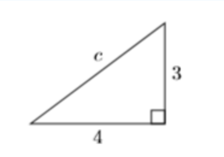
We say “The sum of the squares of the legs of a right triangle equals the square of its hypotenuse.”

Good hint: Note that the hypotenuse sits by itself on one side of the equation $a^2 + b^2 = c^2$. The legs of the hypotenuse are on the other side.

Let’s put the Pythagorean Theorem to work.

Example 8.2.4

Find the length of the missing side of the right triangle shown below.



Solution

First, write out the Pythagorean Theorem, then substitute the given values in the appropriate places.

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean Theorem.} \\ (4)^2 + (3)^2 &= c^2 && \text{Substitute: 4 for } a, 3 \text{ for } b \\ 16 + 9 &= c^2 && \text{Square: } (4)^2 = 16, (3)^2 = 9 \\ 25 &= c^2 && \text{Add: } 16 + 9 = 25 \end{aligned}$$

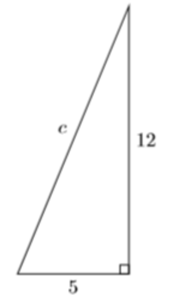
The equation $c^2 = 25$ has two real solutions, $c = -5$ and $c = 5$. However, in this situation, c represents the length of the hypotenuse and must be a positive number. Hence:

$$c = 5 \quad \text{Nonnegative square root.}$$

Thus, the length of the hypotenuse is 5.

Exercise 8.2.4

Find the missing side of the right triangle shown below.



Answer

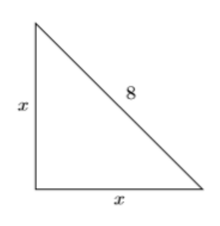
13

Example 8.2.5

An isosceles right triangle has a hypotenuse of length 8. Find the lengths of the legs.

Solution

In general, an isosceles triangle is a triangle with two equal sides. In this case, an isosceles right triangle has two equal legs. We'll let x represent the length of each leg.



Use the Pythagorean Theorem, substituting x for each leg and 8 for the hypotenuse.

$$\begin{aligned} a^2+b^2 &= c^2 && \text{Pythagorean Theorem.} \\ x^2+x^2 &= 8^2 && \text{Substitute: } x \text{ for } a, x \text{ for } b, 8 \text{ for } c. \\ 2x^2 &= 64 && \text{Combine like terms: } x+x=2x \\ x^2 &= 32 && \text{Divide both sides by } 2 \end{aligned}$$

The equation $x^2=32$ has two real solutions, $x=-\sqrt{32}$ and $x=\sqrt{32}$. However, in this situation, x represents the length of each leg and must be a positive number. Hence:

$$x=\sqrt{32} \quad \text{Nonnegative square root.}$$

Remember, your final answer must be in simple radical form. We must factor out a perfect square when possible.

$$\begin{aligned} x &= \sqrt{16}\sqrt{2} && \text{Factor out a perfect square.} \\ x &= 4\sqrt{2} && \text{Simplify: } \sqrt{16}=4 \end{aligned}$$

Thus, the length of each leg is $4\sqrt{2}$.

Exercise 8.2.5

An isosceles right triangle has a hypotenuse of length 10. Find the lengths of the legs.

Answer

Each leg has length $5\sqrt{2}$

Applications

Let’s try a word problem.

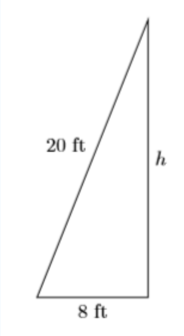
Example 8.2.6

A ladder 20 feet long leans against the garage wall. If the base of the ladder is 8 feet from the garage wall, how high up the garage wall does the ladder reach? Find an exact answer, then use your calculator to round your answer to the nearest tenth of a foot.

Solution

As always, we obey the *Requirements for Word Problem Solutions*.

a. *Set up a variable dictionary.* We’ll create a well-marked diagram for this purpose, letting h represent the distance between the base of the garage wall and the upper tip of the ladder.



b. *Set up an equation.* Using the Pythagorean Theorem, we can write:

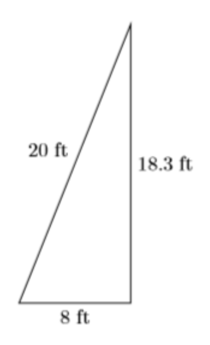
$$\begin{aligned} 8^2 + h^2 &= 20^2 && \text{Pythagorean Theorem.} \\ 64 + h^2 &= 400 && \text{Square: } 8^2 = 64 \text{ and } 20^2 = 400 \end{aligned}$$

c. *Solve the equation.*

$$\begin{aligned} h^2 &= 336 && \text{Subtract 64 from both sides.} \\ h &= \sqrt{336} && h \text{ will be the nonnegative square root.} \\ h &= \sqrt{16}\sqrt{21} && \text{Factor out a perfect square.} \\ h &= 4\sqrt{21} && \text{Simplify: } \sqrt{16} = 4 \end{aligned}$$

d. *Answer the question.* The ladder reaches $4\sqrt{21}$ feet up the wall. Using a calculator, this is about 18.3 feet, rounded to the nearest tenth of a foot.

e. *Look back.* Understand that when we use 18.3 ft, an approximation, our solution will only check approximately.



Using the Pythagorean Theorem:

$$8^2 + 18.3^2 \overset{?}{=} 20^2$$

$$64+334.89 \overset{?}{=} 400$$
$$398.89 \overset{?}{=} 400$$

The approximation is not perfect, but it seems close enough to accept this solution.

Exercise 8.2.6

A ladder 15 feet long leans against a wall. If the base of the ladder is 6 feet from the wall, how high up the wall does the ladder reach? Use your calculator to round your answer to the nearest tenth of a foot.

Answer

13.7 feet.

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8.3: Completing the Square

In Introduction to Radical Notation, we showed how to solve equations such as $x^2 = 9$ both algebraically and graphically.

$$\begin{aligned} x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

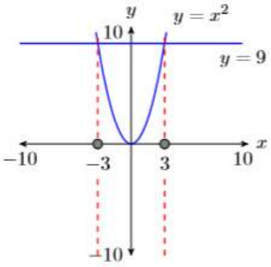


Figure 8.3.1

Note that when we take the square root of both sides of this equation, there are two answers, one negative and one positive.

A perfect square is nice, but not required. Indeed, we may even have to factor out a perfect square to put our final answer in simple form.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ x &= \pm\sqrt{4}\sqrt{2} \\ x &= \pm 2\sqrt{2} \end{aligned}$$

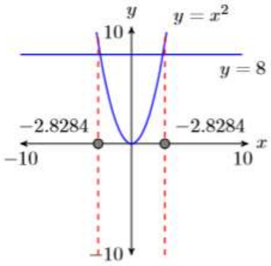


Figure 8.3.2

Readers should use their calculators to check that $-2\sqrt{2} \approx -2.8284$ and $2\sqrt{2} \approx 2.8284$.

Now, let’s extend this solution technique to a broader class of equations.

Example 8.3.1

Solve for $x : (x-4)^2 = 9$

Solution

Much like the solutions of $x^2 = 9$ are $x = \pm 3$, we use a similar approach on $(x-4)^2 = 9$ to obtain:

$$\begin{aligned} (x-4)^2 &= 9 && \text{Original equation.} \\ x-4 &= \pm 3 && \text{There are two square roots.} \end{aligned}$$

To complete the solution, add 4 to both sides of the equation.

$$x = 4 \pm 3 \qquad \text{Add 3 to both sides.}$$

Note that this means that there are two answers, namely:

$$\begin{aligned} x &= 4 - 3 \\ x &= 1 \end{aligned}$$

or

$$\begin{aligned}x &= 4+3\\ x &= 7\end{aligned}$$

Check: Check each solution by substituting it into the original equation.

Substitute 1 for x:

$$\begin{aligned}(x-4)^2 &= 9\\ (1-4)^2 &= 9\\ (-3)^2 &= 9\end{aligned}$$

Substitute 7 for x:

$$\begin{aligned}(x-4)^2 &= 9\\ (7-4)^2 &= 9\\ (3)^2 &= 9\end{aligned}$$

Because the last statement in each check is a true statement, both $x=1$ and $x=7$ are valid solutions of $(x-4)^2=9$.

Exercise $8.3.1$

Solve for $x:(x+6)^2=10$

Answer

$-2,-10$

In Example $8.3.1$, the right-hand side of the equation $(x-4)^2=9$ was a perfect square. However, this is not required, as the next example will show.

Example $8.3.2$

Solve for $x:(x+5)^2=7$

Solution

Using the same technique as in Example $8.3.1$, we obtain:

$$\begin{aligned}(x+5)^2 &= 7 && \text{Original equation.}\\ x+5 &= \pm\sqrt{7} && \text{There are two square roots.}\end{aligned}$$

To complete the solution, subtract 5 from both sides of the equation.

$$x=-5\pm\sqrt{7} \quad \text{Subtract 5 from both sides.}$$

Note that this means that there are two answers, namely:

$$x=-5-\sqrt{7} \quad \text{or} \quad x=-5+\sqrt{7}$$

Check: Check each solution by substituting it into the original equation.

Substitute $-5-\sqrt{7}$ for x:

$$\begin{aligned}(x+5)^2 &= 7\\ ((-5-\sqrt{7})+5)^2 &= 7\\ (-\sqrt{7})^2 &= 7\end{aligned}$$

Substitute $-5+\sqrt{7}$ for x:

$$\begin{aligned}(x+5)^2 &= 7\\ ((-5+\sqrt{7})+5)^2 &= 7\\ (\sqrt{7})^2 &= 7\end{aligned}$$

Because the last statement in each check is a true statement, both $x=-5-\sqrt{7}$ and $x=-5+\sqrt{7}$ are valid solutions of $(x+5)^2 = 7$.

Exercise 8.3.2

Solve for $x : (x-4)^2 = 5$

Answer

$4+\sqrt{5}, 4-\sqrt{5}$

Sometimes you will have to factor out a perfect square to put your answer in simple form.

Example 8.3.3

Solve for $x : (x+4)^2 = 20$

Solution

Using the same technique as in Example 8.3.1, we obtain:

$$\begin{aligned}
(x+4)^2 &= 20 && \text{Original equation.} \\
x+4 &= \pm\sqrt{20} && \text{There are two square roots.} \\
x+4 &= \pm\sqrt{4}\sqrt{5} && \text{Factor out a perfect square.} \\
x+4 &= \pm 2\sqrt{5} && \text{Simplify: } \sqrt{4}=2
\end{aligned}$$

To complete the solution, subtract 4 from both sides of the equation.

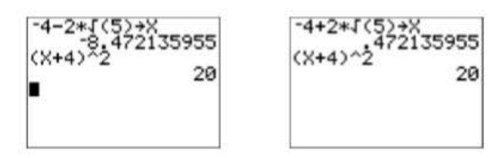
$$x=-4\pm 2\sqrt{5} \quad \text{Subtract 4 from both sides.}$$

Note that this means that there are two answers, namely:

$$x=-4-2\sqrt{5} \quad \text{or} \quad x=-4+2\sqrt{5}$$

Check: Although it is possible to check the exact answers, let's use our calculator instead. First, store $-4-2\sqrt{5}$ in **X**. Next, enter the left-hand side of the equation $(x+4)^2 = 20$ (see image on the left in Figure 8.3.3). Note that (x+4)2 simplifies to 20, showing that $-4-2\sqrt{5}$ is a solution of $(x+4)^2 = 20$.

In similar fashion, the solution $-4+2\sqrt{5}$ also checks in $(x+4)^2 = 20$ (see image on the right in Figure 8.3.3).



```
-4-2*√(5)→X
      -8.472135955
(X+4)^2
                20
```

```
-4+2*√(5)→X
       .472135955
(X+4)^2
                20
```

Figure 8.3.3: Checking $-4-2\sqrt{5}$ and $-4+2\sqrt{5}$ in the equation $(x+4)^2 = 20$.

Exercise 8.3.3

Solve for $x : (x+7)^2 = 18$

Answer

$-7+3\sqrt{2}, -7-3\sqrt{2}$

Perfect Square Trinomials Revisited

Recall the *squaring a binomial* shortcut.

Squaring a Binomial

If a and b are any real numbers, then:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

That is, you square the first term, take the product of the first and second terms and double the result, then square the third term.

Reminder examples:

$$\begin{aligned}(x+3)^2 &= x^2 + 2(x)(3) + 3^2 \\ &= x^2 + 6x + 9\end{aligned}$$

$$\begin{aligned}(x-8)^2 &= x^2 - 2(x)(8) + 8^2 \\ &= x^2 - 16x + 64\end{aligned}$$

Because factoring is "unmultiplying," it is a simple matter to reverse the multiplication process and factor these *perfect square trinomials*.

$$x^2 + 6x + 9 = (x+3)^2$$

$$x^2 - 16x + 64 = (x-8)^2$$

Note how in each case we simply take the square root of the first and last terms.

Example 8.3.4

Factor each of the following trinomials:

a. $x^2 - 12x + 36$
b. $x^2 + 10x + 25$
c. $x^2 - 34x + 289$

Solution

Whenever the first and last terms of a trinomial are perfect squares, we should suspect that we have a perfect square trinomial.

a. The first and third terms of $x^2 - 12x + 36$ are perfect squares. Hence, we take their square roots and try:

$$x^2 - 12x + 36 = (x-6)^2$$

Note that $2(x)(6) = 12x$, which is the middle term on the left. The solution checks.

b. The first and third terms of $x^2 + 10x + 25$ are perfect squares. Hence, we take their square roots and try:

$$x^2 + 10x + 25 = (x+5)^2$$

Note that $2(x)(5) = 10x$, which is the middle term on the left. The solution checks.

c. The first and third terms of $x^2 - 34x + 289$ are perfect squares. Hence, we take their square roots and try:

$$x^2 - 34x + 289 = (x-17)^2$$

Note that $2(x)(17) = 34x$, which is the middle term on the left. The solution checks.

Exercise 8.3.4

Factor: $x^2 + 30x + 225$

Answer

$(x+15)^2$

Completing the Square

In this section we start with the binomial x^2+bx and ask the question "What constant value should we add to x^2+bx so that the resulting trinomial is a perfect square trinomial?" The answer lies in this procedure.

Completing the square

To calculate the constant required to make x^2+bx a perfect square trinomial:

1. Take one-half of the coefficient of $x: \frac{b}{2}$
2. Square the result of step one: $\left(\frac{b}{2}\right)^2=\frac{b^2}{4}$
3. Add the result of step two to $x^2+bx: x^2+bx+\frac{b^2}{4}$

If you follow this process, the result will be a perfect square trinomial which will factor as follows:

$$x^2+bx+\frac{b^2}{4}=\left(x+\frac{b}{2}\right)^2$$

Example 8.3.5

Given x^2+12x, complete the square to create a perfect square trinomial.

Solution

Compare x^2+12x with x^2+bx and note that $b=12$.

1. Take one-half of $12: 6$
2. Square the result of step one: $6^2=36$
3. Add the result of step two to $x^2+12x: x^2+12x+36$

Check: Note that the first and last terms of $x^2+12x+36$ are perfect squares. Take the square roots of the first and last terms and factor as follows:

$$x^2+12x+36=(x+6)^2$$

Note that $2(x)(6)=12x$, so the middle term checks.

Exercise 8.3.5

Given x^2+16x, complete the square to create a perfect square trinomial.

Answer

$x^2+16x+64=(x+8)^2$

Example 8.3.6

Given x^2-3x, complete the square to create a perfect square trinomial.

Solution

Compare x^2-3x with x^2+bx and note that $b=-3$.

1. Take one-half of $-3: -\frac{3}{2}$
2. Square the result of step one: $\left(-\frac{3}{2}\right)^2=\frac{9}{4}$
3. Add the result of step two to $x^2-3x: x^2-3x+\frac{9}{4}$

Check: Note that the first and last terms of $x^2-3x+\frac{9}{4}$ are perfect squares. Take the square roots of the first and last terms and factor as follows:

$$x^2-3x+\frac{9}{4}=\left(x-\frac{3}{2}\right)^2$$

Note that $2(x)\left(\frac{3}{2}\right)=3x$, so the middle term checks.

Exercise 8.3.6

Given x^2-5x, complete the square to create a perfect square trinomial.

Answer

$$x^2-5x+\frac{10}{4}=\left(x-\frac{5}{2}\right)^2$$

Solving Equations by Completing the Square

Consider the following nonlinear equation.

$$x^2=2x+2$$

The standard approach is to make one side zero and factor.

$$x^2-2x-2=0$$

However, one quickly realizes that there is no integer pair whose product is $ac=-2$ and whose sum is $b=-2$. So, what does one do in this situation? The answer is "Complete the square."

Example 8.3.7

Use completing the square to help solve $x^2=2x+2$.

Solution

First, move $2x$ to the left-hand side of the equation, keeping the constant 2 on the right-hand side of the equation.

$$x^2-2x=2$$

On the left, take one-half of the coefficient of $x:\left(\frac{1}{2}\right)(-2)=-1$. Square the result: $(-1)^2=1$. Add this result to both sides of the equation.

$$\begin{aligned}x^2-2x+1&=2+1\\x^2-2x+1&=3\end{aligned}$$

We can now factor the left-hand side as a perfect square trinomial.

$$(x-1)^2=3$$

Now, as in Examples $8.3.1$, $8.3.2$, and $8.3.3$, we can take the square root of both sides of the equation. Remember, there are two square roots.

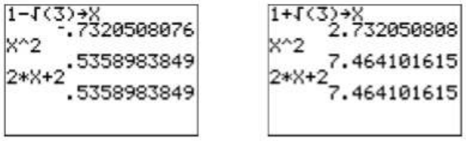
$$x-1=\pm\sqrt{3}$$

Finally, add 1 to both sides of the equation.

$$x=1\pm\sqrt{3}$$

Thus, the equation $x^2=2x+2$ has two answers, $x=1-\sqrt{3}$ and $x=1+\sqrt{3}$.

Check: Let's use the calculator to check the solutions. First, store $1-\sqrt{3}$ in **X** (see the image on the left in Figure $8.3.4$). Then enter the left- and right-hand sides of the equation $x^2=2x+2$ and compare the results (see the image on the left in Figure $8.3.4$). In similar fashion, check the second answer $1+\sqrt{3}$ (see the image on the right in Figure $8.3.4$).



```
1-√(3)→X
          -.7320508076
X^2
           .5358983849
2*X+2
           .5358983849
```

```
1+√(3)→X
           2.732050808
X^2
           7.464101615
2*X+2
           7.464101615
```

Figure $8.3.4$: Checking $1-\sqrt{3}$ and $1+\sqrt{3}$ in the equation $x^2 = 2x + 2$.

In both cases, note that the left- and right-hand sides of $x^2 = 2x + 2$ produce the same result. Hence, both $1-\sqrt{3}$ and $1+\sqrt{3}$ are valid solutions of $x^2 = 2x + 2$.

Exercise $8.3.7$

Use completing the square to help solve $x^2 = 3 - 6x$.

Answer

$-3+2\sqrt{3}, -3-2\sqrt{3}$

Example $8.3.8$

Solve the equation $x^2 - 8x - 12 = 0$, both algebraically and graphically. Compare your answer from each method.

Solution

First, move the constant 12 to the right-hand side of the equation.

$$\begin{aligned} x^2 - 8x - 12 &= 0 &&\text{Original equation.} \\ x^2 - 8x &= 12 &&\text{Add 12 to both sides.} \end{aligned}$$

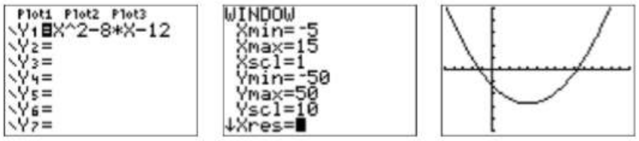
Take half of the coefficient of $x : (1/2)(-8) = -4$. Square: $(-4)^2 = 16$. Now add 16 to both sides of the equation.

$$\begin{aligned} x^2 - 8x + 16 &= 12 + 16 &&\text{Add 16 to both sides.} \\ (x-4)^2 &= 28 &&\text{Factor left-hand side.} \\ x - 4 &= \pm\sqrt{28} &&\text{There are two square roots.} \end{aligned}$$

Note that the answer is not in simple radical form.

$$\begin{aligned} x - 4 &= \pm\sqrt{4}\sqrt{7} &&\text{Factor out a perfect square.} \\ x - 4 &= \pm 2\sqrt{7} &&\text{Simplify: } \sqrt{4} = 2 \\ x &= 4 \pm 2\sqrt{7} &&\text{Add 4 to both sides.} \end{aligned}$$

Graphical solution: Enter the equation $y = x^2 - 8x - 12$ in **Y1** of the **Y= menu** (see the first image in Figure $8.3.5$). After some experimentation, we settled on the **WINDOW** parameters shown in the middle image of Figure $8.3.5$. Once you've entered these **WINDOW** parameters, push the **GRAPH** button to produce the rightmost image in Figure $8.3.5$.



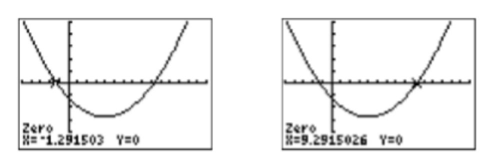
```
 Plot1 Plot2 Plot3
\Y1■X^2-8*X-12
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
 Xmin=-5
 Xmax=15
 Xscl=1
 Ymin=-50
 Ymax=50
 Yscl=10
↓Xres=■
```

Figure $8.3.5$: Drawing the graph of $y = x^2 - 8x - 12$.

We're looking for solutions of $x^2 - 8x - 12 = 0$, so we need to locate where the graph of $y = x^2 - 8x - 12$ intercepts the x-axis. That is, we need to find the zeros of $y = x^2 - 8x - 12$. Select **2:zero** from the **CALC** menu, move the cursor slightly to the left of the first x-intercept and press **ENTER** in response to "Left bound." Move the cursor slightly to the right of the first x-intercept and press **ENTER** in response to "Right bound." Leave the cursor where it sits and press **ENTER** in response to "Guess." The calculator responds by finding the x-coordinate of the x-intercept, as shown in the first image in Figure $8.3.6$.

Repeat the process to find the second x-intercept of $y = x^2 - 8x - 12$ shown in the second image in Figure $8.3.6$.



```
Zero
X=-1.291503  Y=0
```

```
Zero
X=9.2915026  Y=0
```

Figure 8.3.6: Calculating the zeros of $y=x^2-8x-12$.

Reporting the solution on your homework: Duplicate the image in your calculator's viewing window on your homework page. Use a ruler to draw all lines, but freehand any curves.

- Label the horizontal and vertical axes with x and y, respectively (see Figure $8.3.7$).
- Place your **WINDOW** parameters at the end of each axis (see Figure $8.3.7$).
- Label the graph with its equation (see Figure $8.3.7$).
- Drop dashed vertical lines through each x-intercept. Shade and label the x-values of the points where the dashed vertical line crosses the x-axis. These are the solutions of the equation $x^2-8x-12=0$ (see Figure $8.3.7$).

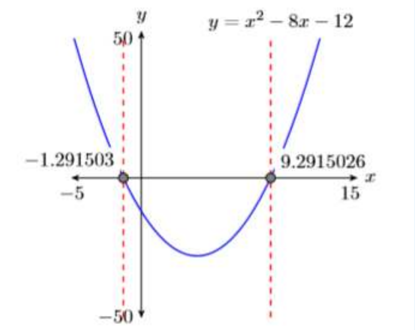
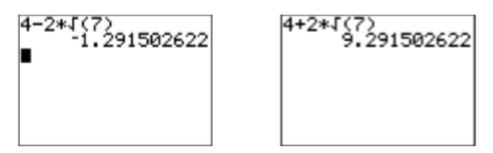


Figure $8.3.7$: Reporting your graphical solution on your homework.

Thus, the graphing calculator reports that the solutions of $x^2-8x-12=0$ are $x\approx-1.291503$ and $x\approx 9.2915026$.

Comparing exact and calculator approximations: How well do the graphing calculator solutions compare with the exact solutions, $x=4-2\sqrt{7}$ and $x=4+2\sqrt{7}$? After entering each in the calculator (see Figure $8.3.8$), the comparison is excellent!



```
4-2*√(7)
        -1.291502622
```

```
4+2*√(7)
         9.291502622
```

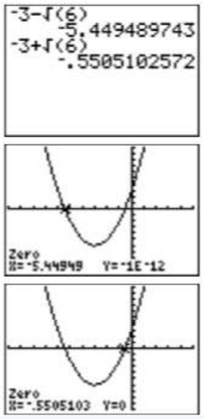
Figure $8.3.8$: Approximating exact solutions $x=4-2\sqrt{7}$ and $x=4+2\sqrt{7}$.

Exercise $8.3.8$

Solve the equation $x^2+6x+3=0$ both algebraically and graphically, then compare your answers.

Answer

$-3-\sqrt{6},-3+\sqrt{6}$



```
-3-√(6)
         -5.449489743
-3+√(6)
         -.5505102572
```

```
Zero
X=-5.44949    Y=-1E-12
```

```
Zero
X=-.5505103  Y=0
```

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8.4: The Quadratic Formula

We first start with the definition of a *quadratic equation*.

Quadratic equation

A second degree polynomial equation of the form

$$ax^2+bx+c=0$$

where a, b, and c are any real numbers, is called a quadratic equation in x.

The goal of this section is to develop a formulaic shortcut that will provide exact solutions of the quadratic equation ax2 +bx+c = 0. We start by moving the constant term to the other side of the equation.

$$\begin{aligned} ax^2+bx+c &= 0 && \text{Quadratic equation.} \\ ax^2+bx &= -c && \text{Subtract } c \text{ from both sides.} \end{aligned}$$

In preparation for completing the square, we next divide both sides of the equation by a.

$$x^2+\frac{b}{a}x=-\frac{c}{a} \quad \text{Divide both sides by } a$$

Now we complete the square. Take one-half of the coefficient of x, then square the result.

$\frac{1}{2}\cdot\frac{b}{a}=\frac{b}{2a}$ when squared gives $\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$

We now add $\frac{b^2}{4a^2}$ to both sides of the equation.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2} \quad \text{Add } b^2/\left(4a^2\right) \text{ to both sides.}$$

On the left, we factor the perfect square trinomial. On the right, we make equivalent fractions with a common denominator.

$$\begin{aligned} \left(x+\frac{b}{2a}\right)^2 &= -\frac{c}{a}\cdot\frac{4a}{4a}+\frac{b^2}{4a^2} && \text{On the left, factor. On the right,} \\ & && \text{create equivalent fractions with} \\ \left(x+\frac{b}{2a}\right)^2 &= -\frac{4ac}{4a^2}+\frac{b^2}{4a^2} && \text{Multiply numerators and denominators.} \\ \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} && \text{Add fractions.} \end{aligned}$$

When we take the square root, there are two answers.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \quad \text{Two square roots.}$$

When you take the square root of a fraction, you take the square root of both the numerator and denominator.

$$\begin{aligned} x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}} \\ x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} && \text{Simplify: } \sqrt{4a^2}=2a \\ x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} && \text{Subtract } b/(2a) \text{ from both sides} \end{aligned}$$

Because both fractions have the same denominator, we can add and subtract numerators and put the answer over the common denominator.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The quadratic formula

The equation $ax^2+bx+c=0$ is called a quadratic equation. Its solutions are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

called the *quadratic formula*.

Whew! Fortunately, the result is a lot easier to apply than it is to develop! Let’s try some examples.

Example $8.4.1$

Solve for $x: x^2-4x-5=0$

Solution

The integer pair $1,-5$ has product $ac=-5$ and sum $b=-4$. Hence, this trinomial factors.

$$\begin{aligned} x^2-4x-5&=0\\ (x+1)(x-5)&=0 \end{aligned}$$

Now we can use the zero product property to write:

$$\begin{aligned} x+1&=0 &\quad \text{or} \quad& x-5&=0\\ x&=-1 && x&=5 \end{aligned}$$

Thus, the solutions are $x=-1$ and $x=5$. Now, let’s give the quadratic formula a try. First, we must compare our equation with the quadratic equation, then determine the values of a, b, and c.

$$\begin{aligned} ax^2+bx+c=0\\ x^2-4x-5=0 \end{aligned}$$

Comparing equations, we see that $a=1$, $b=-4$, and $c=-5$. We will now plug these numbers into the quadratic formula. First, replace each occurrence of a, b, and c in the quadratic formula with open parentheses.

$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a} && \color{red}{\text{The quadratic formula.}}\\ x&=\frac{-(\quad)\pm\sqrt{(\quad)^2-4()()}}{2()} && \color{red}{\text{Replace } a, b, \text{ and } c \text{ with open parentheses.}} \end{aligned}$$

Now we can substitute: 1 for a, -4 for b, and -5 for$c$.

$$\begin{aligned} x&=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-5)}}{2(1)} && \color{red}{\text{Substitute: } 1 \text{ for } a, -4 \text{ for } b}\\ x&=\frac{4\pm\sqrt{16+20}}{2} && \color{red}{\text{Simplify. Exponent first, then}}\\ x&=\frac{4\pm\sqrt{36}}{2} && \color{red}{\text{Add: } 16+20=36}\\ x&=\frac{4\pm 6}{2} && \color{red}{\text{Simplify: } \sqrt{36}=6} \end{aligned}$$

Note that because of the “plus or minus” symbol, we have two answers.

$$\begin{aligned} x&=\frac{4-6}{2} &\quad \text{or} \quad& x=\frac{4+6}{2}\\ x&=\frac{-2}{2} && x=\frac{10}{2}\\ x&=-1 && x=5 \end{aligned}$$

Note that these answers match the answers found using the ac-test to factor the trinomial.

Exercise 8.4.1

Solve for $x: x^2 - 8x + 12 = 0$

Answer

2, 6

Example 8.4.2

Solve for $x: x^2 = 5x + 7$

Solution

The equation is nonlinear, make one side zero.

$$\begin{aligned} x^2 &= 5x + 7 && \text{Original equation.} \\ x^2 - 5x - 7 &= 0 && \text{Nonlinear. Make one side zero.} \end{aligned}$$

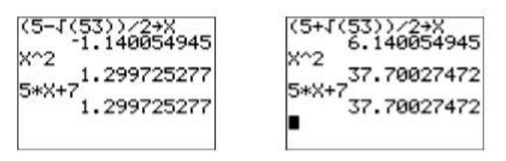
Compare $x^2 - 5x - 7 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -5$, and $c = -7$. Replace each occurrence of a, b, and c with open parentheses to prepare the quadratic formula for substitution.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula.} \\ x &= \frac{-() \pm \sqrt{()^2 - 4()()}}{2()} && \text{Replace } a, b, \text{ and } c \text{ with} \end{aligned}$$

Substitute 1 for a, -5 for b, and -7 for c.

$$\begin{aligned} x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-7)}}{2(1)} && \text{Substitute: } a = 1, b = -5, c = -7 \\ x &= \frac{5 \pm \sqrt{25 + 28}}{2} && \text{Exponents and multiplication first.} \\ x &= \frac{5 \pm \sqrt{53}}{2} && \text{Simplify.} \end{aligned}$$

Check: Use the calculator to check each solution (see Figure 8.4.1). Note that in storing $(5 - \sqrt{53})/2$ in **X**, we must surround the numerator in parentheses.



```
(5-√(53))/2→X
         -1.140054945
X^2
          1.299725277
5*X+7
          1.299725277
```

```
(5+√(53))/2→X
          6.140054945
X^2
          37.70027472
5*X+7
          37.70027472
■
```

***Figure** 8.4.1: Check $(5 - \sqrt{53})/2$ and $(5 + \sqrt{53})/2$.*

In each image in Figure 8.4.1, after storing the solution in **X**, note that the left and right-hand sides of the original equation $x^2 = 5x + 7$ produce the same number, verifying that our solutions are correct.

Exercise 8.4.2

Solve for $x: x^2 + 7x = 10$

Answer

$(-7 + \sqrt{89})/2, (-7 - \sqrt{89})/2$

In addition to placing all square roots into simple radical form, sometimes you need to reduce your answer to lowest terms.

Example 8.4.3

Solve for $x : 7x^2 - 10x + 1 = 0$

Solution

Compare $7x^2 - 10x + 1 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 7$, $b = -10$, and $c = 1$. Replace each occurrence of a, b, and c with open parentheses to prepare the quadratic formula for substitution.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \color{red}{\text{The quadratic formula.}} \\ x &= \frac{-(\quad) \pm \sqrt{()^2 - 4()()}}{2()} && \color{red}{\text{Replace } a, b, \text{ and } c \text{ with open parentheses.}} \end{aligned}$$

Substitute 7 for a, -10 for b, and 1 for c.

$$\begin{aligned} x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(7)(1)}}{2(7)} && \color{red}{\text{Substitute: 7 for } a} \\ x &= \frac{10 \pm \sqrt{100 - 28}}{14} && \color{red}{\text{Exponent, then multiplication.}} \\ x &= \frac{10 \pm \sqrt{72}}{14} && \color{red}{\text{Simplify.}} \end{aligned}$$

In this case, note that we can factor out a perfect square, namely $\sqrt{36}$.

$$\begin{aligned} x &= \frac{10 \pm \sqrt{36}\sqrt{2}}{14} && \color{red}{\sqrt{72} = \sqrt{36}\sqrt{2}} \\ x &= \frac{10 \pm 6\sqrt{2}}{14} && \color{red}{\text{Simplify: } \sqrt{36} = 6} \end{aligned}$$

Finally, notice that both numerator and denominator are divisible by 2.

$$\begin{aligned} x &= \frac{\frac{10 \pm 6\sqrt{2}}{2}}{\frac{14}{2}} && \color{red}{\text{Divide numerator and denominator by 2.}} \\ x &= \frac{\frac{10}{2} \pm \frac{6\sqrt{2}}{2}}{\frac{14}{2}} && \color{red}{\text{Distribute the 2.}} \\ x &= \frac{5 \pm 3\sqrt{2}}{7} && \color{red}{\text{Simplify.}} \end{aligned}$$

Alternate simplification: Rather than dividing numerator and denominator by 2, some prefer to factor and cancel, as follows.

$$\begin{aligned} x &= \frac{10 \pm 6\sqrt{2}}{14} && \color{red}{\text{Original answer.}} \\ x &= \frac{2(5 \pm 3\sqrt{2})}{2(7)} && \color{red}{\text{Factor out a 2}} \\ x &= \frac{\cancel{2}(5 \pm 3\sqrt{2})}{\cancel{2}(7)} && \color{red}{\text{Cancel.}} \\ x &= \frac{5 \pm 3\sqrt{2}}{7} && \color{red}{\text{Simplify.}} \end{aligned}$$

Note that we get the same answer using this technique.

Exercise 8.4.3

Solve for $x : 3x^2 + 8x + 2 = 0$

Answer

$(-4 + \sqrt{10})/3, (-4 - \sqrt{10})/3$

Example 8.4.4

An object is launched vertically and its height y (in feet) above ground level is given by the equation $y = 320 + 192t - 16t^2$, wheret is the time (in seconds) that has passed since its launch. How much time must pass after the launch before the object returns to ground level? After placing the answer in simple form and reducing, use your calculator to round the answer to the nearest tenth of a second.

Solution

When the object returns to ground level, its height y above ground level is $y = 0$ feet. To find the time when this occurs, substitute $y = 0$ in the formula $y = 320 + 192t - 16t^2$ and solve for t.

$$\begin{aligned} y &= 320 + 192t - 16t^2 && \text{Original equation.} \\ 0 &= 320 + 192t - 16t^2 && \text{Set } y = 0 \end{aligned}$$

Each of the coefficients is divisible by -16.

$$0 = t^2 - 12t - 20 \qquad \text{Divide both sides by } -16$$

Compare $t^2 - 12t - 20 = 0$ with $at^2 + bt + c = 0$ and note that $a = 1$, $b = -12$, and $c = -20$. Replace each occurrence of a, b, and c with open parentheses to prepare the quadratic formula for substitution. Note that we are solving for t this time, not x.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula.} \\ x &= \frac{-() \pm \sqrt{()^2 - 4()()}}{2()} && \text{Replace } a, b, \text{ and } c \text{ with open parentheses.} \end{aligned}$$

Substitute 1 for a, -12 for b, and -20 for c.

$$\begin{aligned} t &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(-20)}}{2(1)} && \text{Substitute: 1 for } a \\ t &= \frac{12 \pm \sqrt{144 + 80}}{2} && \text{Exponent, then multiplication.} \\ t &= \frac{12 \pm \sqrt{224}}{2} && \text{Simplify.} \end{aligned}$$

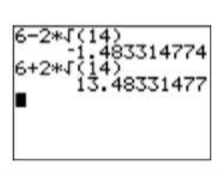
The answer is not in simple form, as we can factor out $\sqrt{16}$.

$$\begin{aligned} t &= \frac{12 \pm \sqrt{16}\sqrt{14}}{2} && \sqrt{224} = \sqrt{16}\sqrt{14} \\ t &= \frac{12 \pm 4\sqrt{14}}{2} && \text{Simplify: } \sqrt{16} = 4 \end{aligned}$$

Use the distributive property to divide both terms in the numerator by 2.

$$\begin{aligned} t &= \frac{12}{2} \pm \frac{4\sqrt{14}}{2} && \text{Divide both terms by 2} \\ t &= 6 \pm 2\sqrt{14} && \text{Simplify} \end{aligned}$$

Thus, we have two solutions, $t = 6 - 2\sqrt{14}$ and $t = 6 + 2\sqrt{14}$. Use your calculator to find decimal approximations, then round to the nearest tenth.



```
6-2*√(14)
     -1.483314774
6+2*√(14)
      13.48331477
```

Figure 8.4.2***: Using calculator to find decimal approximations***

$$t \approx -1.5, 13.5$$

The negative time is irrelevant, so to the nearest tenth of a second, it takes the object approximately 13.5 seconds to return to ground level.

Exercise $8.4.4$

An object is launched vertically and its height y (in feet) above ground level is given by the equation $y=160+96t-16t^2$, where t is the time (in seconds) that has passed since its launch. How much time must pass after the launch before the object returns to ground level?

Answer

$3+\sqrt{19} \approx 7.4$ seconds

Example $8.4.5$

Arnie gets on his bike at noon and begins to ride due north at a constant rate of 12 miles per hour. At 1:00 PM, Barbara gets on her bike at the same starting point and begins to ride due east at a constant rate of 8 miles per hour. At what time of the day will they be 50 miles apart (as the crow flies)? Don't worry about simple form, just report the time of day, correct to the nearest minute.

Solution

At the moment they are 50 miles apart, let t represent the time that Arnie has been riding since noon. Because Barbara started at 1:00 PM, she has been riding for one hour less than Arnie. So, let $t-1$ represent the numbers of hours that Barbara has been riding at the moment they are 50 miles apart.

Now, if Arnie has been riding at a constant rate of 12 miles per hour for t hours, then he has traveled a distance of $12t$ miles. Because Barbara has been riding at a constant rate of 8 miles per hour for $t-1$ hours, she has traveled a distance of $8(t-1)$ miles.

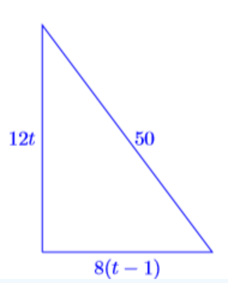


Figure $8.4.3$***:*** 50 *miles apart.*

The distance and direction traveled by Arnie and Barbara are marked in Figure $8.4.3$. Note that we have a right triangle, so the sides of the triangle must satisfy the Pythagorean Theorem. That is,

$$(12t)^2+[8(t-1)]^2=50^2 \qquad \text{Use the Pythagorean Theorem.}$$

Distribute the 8.

$$(12t)^2+(8t-8)^2=50^2 \qquad \text{Distribute the 8}$$

Square each term. Use $(a-b)^2=a^2-2ab+b^2$ to expand $(8t-8)^2$.

$$\begin{aligned} 144t^2+64t^2-128t+64=2500 &\qquad \text{Square each term.} \\ 208t^2-128t+64=2500 &\qquad \text{Simplify: } 144t^2+64t^2=208t^2 \end{aligned}$$

The resulting equation is nonlinear. Make one side equal to zero.

$$\begin{aligned} 208t^2-128t-2436 &=0 \qquad \text{Subtract 2500 from both sides.} \\ 52t^2-32t-609 &=0 \qquad \text{Divide both sides by 4.} \end{aligned}$$

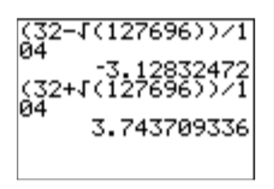
Compare $52t^2 - 32t - 609 = 0$ with $at^2 + bt + c = 0$ and note that $a = 52$, $b = -32$, and $c = -609$. Replace each occurrence of a, b, and c with open parentheses to prepare the quadratic formula for substitution. Note that we are solving for t this time, not x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{The quadratic formula.}$$
$$x = \frac{-() \pm \sqrt{()^2 - 4()()}}{2()} \quad \text{Replace } a, b, \text{ and } c \text{ with}$$

Substitute 52 for a, -32 for b, and -609 for c.

$$t = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(52)(-609)}}{2(52)} \quad \text{Substitute: 52 for } a$$
$$t = \frac{32 \pm \sqrt{1024 + 126672}}{104} \quad \text{Exponent, then multiplication.}$$
$$t = \frac{32 \pm \sqrt{127696}}{104} \quad \text{Simplify.}$$

Now, as the request is for an approximate time, we won't bother with simple form and reduction, but proceed immediately to the calculator to approximate this last result (see Figure $8.4.4$). Thus, Arnie has been riding for approximately 3.743709336 hours. To change the fractional part 0.743709336 hours to minutes, multiply by 60min/hr.



```
(32-√(127696))/1
04
         -3.12832472
(32+√(127696))/1
04
         3.743709336
```

Figure $8.4.4$***:*** *Approximate time that Arnie has been riding.*

$$0.743709336\text{hr} = 0.743709336\text{hr} \times \frac{60\text{min}}{\text{hr}} = 44.62256016\text{min}$$

Rounding to the nearest minute, Arnie has been riding for approximately 3 hours and 45 minutes. Because Arnie started riding at noon, the time at which he and Barbara are 50 miles apart is approximately 3:45 PM.

Exercise $8.4.5$

At 6:00 AM, a freight train passes through Sagebrush Junction heading west at 40 miles per hour. At 8:00 AM, a passenger train passes through the junction heading south at 60 miles per hour. At what time of the day, correct to the nearest minute, will the two trains be 180 miles apart?

Answer

9:42 AM

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8.E: Quadratic Functions (Exercises)

8.1: Introduction to Radical Notation

1) List all real square roots of -400.

Answer

There are no real square roots.

2) List all real square roots of 64.

3) List all real square roots of -25.

Answer

There are no real square roots.

4) List all real square roots of 81.

5) List all real square roots of 49.

Answer

$-7, 7$

6) List all real square roots of -100.

7) List all real square roots of 324.

Answer

$-18, 18$

8) List all real square roots of 36.

9) List all real square roots of -225.

Answer

There are no real square roots.

10) List all real square roots of 0.

11) List all real solutions of $x^2 = -225$.

Answer

There are no real solutions.

12) List all real solutions of $x^2 = -25$.

13) List all real solutions of $x^2 = 361$.

Answer

$-19, 19$

14) List all real solutions of $x^2 = 256$.

15) List all real solutions of $x^2 = -400$.

Answer

There are no real solutions.

16) List all real solutions of $x^2 = 0$.

17) List all real solutions of $x^2 = 169$.

Answer

$-13, 13$

18) List all real solutions of $x^2 = -100$.

19) List all real solutions of $x^2 = 625$.

Answer

$-25, 25$

20) List all real solutions of $x^2 = 324$.

In Exercises 21-30, simplify each of the given expressions.

21) $\sqrt{64}$

Answer

8

22) $-\sqrt{-529}$

23) $-\sqrt{-256}$

Answer

The expression is not a real number.

24) $\sqrt{-529}$

25) $-\sqrt{361}$

Answer

-19

26) $\sqrt{-361}$

27) $-\sqrt{100}$

Answer

-10

28) $-\sqrt{196}$

29) $\sqrt{441}$

Answer

21

30) $\sqrt{49}$

In Exercises 31-38, simplify each of the given expressions.

31) $(-\sqrt{17})^2$

Answer

17

32) $(-\sqrt{31})^2$

33) $(\sqrt{59})^2$

Answer

59

34) $(\sqrt{43})^2$

35) $(-\sqrt{29})^2$

Answer

29

36) $(-\sqrt{89})^2$

37) $(\sqrt{79})^2$

Answer

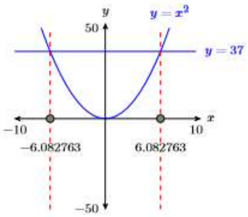
79

38) $(\sqrt{3})^2$

In Exercises 39-42, for each of the given equations, first use the 5:intersect utility on the CALC menu of the graphing calculator to determine the solutions. Follow the Calculator Submission Guidelines, as demonstrated in Example 8.1.9 in reporting the solution on your homework paper. Second, solve the equation algebraically, then use your calculator to find approximations of your answers and compare this second set with the first set of answers.

39) $x^2 = 37$

Answer

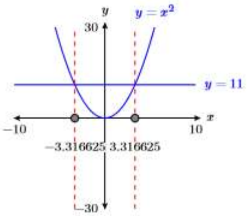


$\pm\sqrt{37} \approx \pm 6.082763$

40) $x^2 = 32$

41) $x^2 = 11$

Answer



$\pm\sqrt{11} \approx \pm 3.316625$

42) $x^2 = 42$

8.2: Simplifying Radical Expressions

In Exercises 1-6, simplify the given expression, writing your answer using a single square root symbol. Check the result with your graphing calculator.

1) $\sqrt{5}\sqrt{13}$

Answer

$\sqrt{65}$

2) $\sqrt{2}\sqrt{7}$

3) $\sqrt{17}\sqrt{2}$

Answer

$\sqrt{34}$

4) $\sqrt{5}\sqrt{11}$

5) $\sqrt{5}\sqrt{17}$

Answer

$\sqrt{85}$

6) $\sqrt{17}\sqrt{3}$

In Exercises 7-26, convert each of the given expressions to simple radical form.

7) $\sqrt{56}$

Answer

$2\sqrt{14}$

8) $\sqrt{45}$

9) $\sqrt{99}$

Answer

$3\sqrt{11}$

10) $\sqrt{75}$

11) $\sqrt{150}$

Answer

$5\sqrt{6}$

12) $\sqrt{90}$

13) $\sqrt{40}$

Answer

$2\sqrt{10}$

14) $\sqrt{171}$

15) $\sqrt{28}$

Answer

$2\sqrt{7}$

16) $\sqrt{175}$

17) $\sqrt{153}$

Answer

$3\sqrt{17}$

18) $\sqrt{125}$

19) $\sqrt{50}$

Answer

$5\sqrt{2}$

20) $\sqrt{88}$

21) $\sqrt{18}$

Answer

$3\sqrt{2}$

22) $\sqrt{117}$

23) $\sqrt{44}$

Answer

$2\sqrt{11}$

24) $\sqrt{20}$

25) $\sqrt{104}$

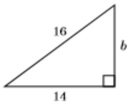
Answer

$2\sqrt{26}$

26) $\sqrt{27}$

In Exercises 27-34, find the length of the missing side of the right triangle. Your final answer must be in simple radical form.

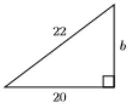
27)



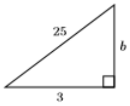
Answer

$2\sqrt{15}$

28)



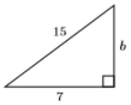
29)



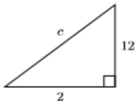
Answer

$2\sqrt{154}$

30)



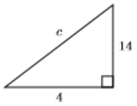
31)



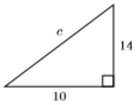
Answer

$2\sqrt{37}$

32)



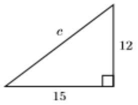
33)



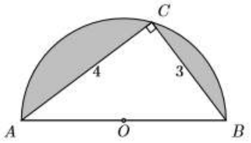
Answer

$2\sqrt{74}$

34)



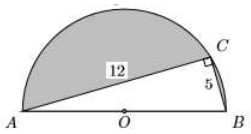
35) In the figure below, a right triangle is inscribed in a semicircle. What is the area of the shaded region?



Answer

$\frac{25}{8}\pi - 6$

36) In the figure below, a right triangle is inscribed in a semicircle. What is the area of the shaded region?



37) The longest leg of a right triangle is 10 feet longer than twice the length of its shorter leg. The hypotenuse is 4 feet longer than three times the length of the shorter leg. Find the lengths of all three sides of the right triangle.

Answer

$7, 24, 25$

38) The longest leg of a right triangle is 2 feet longer than twice the length of its shorter leg. The hypotenuse is 3 feet longer than twice the length of the shorter leg. Find the lengths of all three sides of the right triangle.

39) A ladder 19 feet long leans against the garage wall. If the base of the ladder is 5 feet from the garage wall, how high up the garage wall does the ladder reach? Use your calculator to round your answer to the nearest tenth of a foot.

Answer

18.3 feet

40) A ladder 19 feet long leans against the garage wall. If the base of the ladder is 6 feet from the garage wall, how high up the garage wall does the ladder reach? Use your calculator to round your answer to the nearest tenth of a foot.

8.3: Completing the Square

In Exercises 1-8, find all real solutions of the given equation. Place your final answers in simple radical form.

1) $x^2 = 84$

Answer

$\pm 2\sqrt{21}$

2) $x^2 = 88$

3) $x^2 = 68$

Answer

$\pm 2\sqrt{17}$

4) $x^2 = 112$

5) $x^2 = -16$

Answer

No real solutions

6) $x^2 = -104$

7) $x^2 = 124$

Answer

$\pm 2\sqrt{31}$

8) $x^2 = 148$

In Exercises 9-12, find all real solutions of the given equation. Place your final answers in simple radical form.

9) $(x+19)^2 = 36$

Answer

$-25, -13$

10) $(x-4)^2 = 400$

11) $(x+14)^2 = 100$

Answer

$-24, -4$

12) $(x-15)^2=100$

In Exercises 13-18, square each of the following binomials.

13) $(x+23)^2$

Answer

$x^2+46x+529$

14) $(x-5)^2$

15) $(x+11)^2$

Answer

$x^2+22x+121$

16) $(x-7)^2$

17) $(x-25)^2$

Answer

$x^2-50x+625$

18) $(x+4)^2$

In Exercises 19-24, factor each of the following trinomials.

19) $x^2+24x+144$

Answer:

$$(x+12)^2$$

20) $x^2-16x+64$

21) $x^2-34x+289$

Answer:

$$(x-17)^2$$

22) $x^2+8x+16$

23) $x^2-20x+100$

Answer:

$$(x-10)^2$$

24) $x^2+16x+64$

In Exercises 25-36, for each expression, complete the square to form a perfect square trinomial. Check your answer by factoring your result. Be sure to check your middle term.

25) x^2-20x

Answer

$x^2-20x+100$

26) x^2-10x

27) x^2-6x

Answer

x^2-6x+9

28) x^2-40x

29) x^2+20x

Answer

$x^2+20x+100$

30) x^2+26x

31) x^2+7x

Answer

$x^2+7x+\frac{49}{4}$

32) x^2+19x

33) x^2+15x

Answer

$x^2+15x+\frac{225}{4}$

34) x^2+25x

35) x^2-5x

Answer

$x^2-5x+\frac{25}{4}$

36) x^2-3x

In Exercises 37-52, find all real solutions, if any, of the given equation. Place your final answers in simple radical form.

37) $x^2=18x-18$

Answer

$9-3\sqrt{7}, 9+3\sqrt{7}$

38) $x^2=12x-18$

39) $x^2=16x-16$

Answer

$8-4\sqrt{3}, 8+4\sqrt{3}$

40) $x^2=12x-4$

41) $x^2=-16x-4$

Answer

$-8-2\sqrt{15}, -8+2\sqrt{15}$

42) $x^2=-12x-12$

43) $x^2=18x-9$

Answer

$9-6\sqrt{2}, 9+6\sqrt{2}$

44) $x^2 = 16x - 10$

45) $x^2 = 16x - 8$

Answer

$8 - 2\sqrt{14}, 8 + 2\sqrt{14}$

46) $x^2 = 10x - 5$

47) $x^2 = -18x - 18$

Answer

$-9 - 3\sqrt{7}, -9 + 3\sqrt{7}$

48) $x^2 = -10x - 17$

49) $x^2 = -16x - 20$

Answer

$-8 - 2\sqrt{11}, -8 + 2\sqrt{11}$

50) $x^2 = -16x - 12$

51) $x^2 = -18x - 1$

Answer

$-9 - 4\sqrt{5}, -9 + 4\sqrt{5}$

52) $x^2 = -12x - 8$

In Exercises 53-56, solve the given equation algebraically, stating your final answers in simple radical form. Next, use the graphing calculator to solve the equation, following the technique outlined in Example 8.3.8. Use the Calculator Submission Guidelines, as demonstrated inExample 8, when reporting the solution on your homework. Compare the solutions determined by the two methods.

53) $x^2 - 2x - 17 = 0$

Answer

$1 - 3\sqrt{2}, 1 + 3\sqrt{2}$

54) $x^2 - 4x - 14 = 0$

55) $x^2 - 6x - 3 = 0$

Answer

$3 - 2\sqrt{3}, 3 + 2\sqrt{3}$

56) $x^2 - 4x - 16 = 0$

8.4: The Quadratic Formula

In Exercises 1-8, solve the given equation by factoring the trinomial using the ac-method, then applying the zero product property. Secondly, craft a second solution using the quadratic formula. Compare your answers.

1) $x^2 - 3x - 28 = 0$

Answer

$-4, 7$

2) $x^2 - 4x - 12 = 0$

3) $x^2-8x+15=0$

Answer

$3, 5$

4) $x^2-6x+8=0$

5) $x^2-2x-48=0$

Answer

$-6, 8$

6) $x^2+9x+8=0$

7) $x^2+x-30=0$

Answer

$-6, 5$

8) $x^2-17x+72=0$

In Exercises 9-16, use the quadratic formula to solve the given equation. Your final answers must be reduced to lowest terms and all radical expressions must be in simple radical form.

9) $x^2-7x-5=0$

Answer

$$\frac{7 \pm \sqrt{69}}{2}$$

10) $3x^2-3x-4=0$

11) $2x^2+x-4=0$

Answer

$$\frac{-1 \pm \sqrt{33}}{4}$$

12) $2x^2+7x-3=0$

13) $x^2-7x-4=0$

Answer

$$\frac{7 \pm \sqrt{65}}{2}$$

14) $x^2-5x+1=0$

15) $4x^2-x-2=0$

Answer

$$\frac{1 \pm \sqrt{33}}{8}$$

16) $5x^2+x-2=0$

In Exercises 17-24, use the quadratic formula to solve the given equation. Your final answers must be reduced to lowest terms and all radical expressions must be in simple radical form.

17) $x^2-x-11=0$

Answer

$$\frac{1 \pm 3\sqrt{5}}{2}$$

18) $x^2 - 11x + 19 = 0$

19) $x^2 - 9x + 9 = 0$

Answer

$$\frac{9 \pm 3\sqrt{5}}{2}$$

20) $x^2 + 5x - 5 = 0$

21) $x^2 - 3x - 9 = 0$

Answer

$$\frac{3 \pm 3\sqrt{5}}{2}$$

22) $x^2 - 5x - 5 = 0$

23) $x^2 - 7x - 19 = 0$

Answer

$$\frac{7 \pm 5\sqrt{5}}{2}$$

24) $x^2 + 13x + 4 = 0$

In Exercises 25-32, use the quadratic formula to solve the given equation. Your final answers must be reduced to lowest terms and all radical expressions must be in simple radical form.

25) $12x^2 + 10x - 1 = 0$

Answer

$$\frac{-5 \pm \sqrt{37}}{12}$$

26) $7x^2 + 6x - 3 = 0$

27) $7x^2 - 10x + 1 = 0$

Answer

$$\frac{5 \pm 3\sqrt{2}}{7}$$

28) $7x^2 + 4x - 1 = 0$

29) $2x^2 - 12x + 3 = 0$

Answer

$$\frac{6 \pm \sqrt{30}}{2}$$

30) $2x^2 - 6x - 13 = 0$

31) $13x^2 - 2x - 2 = 0$

Answer

$$\frac{1 \pm 3\sqrt{3}}{13}$$

32) $9x^2 - 2x - 3 = 0$

33) An object is launched vertically and its height y (in feet) above ground level is given by the equation $y = 240 + 160t - 16t^2$, where t is the time (in seconds) that has passed since its launch. How much time must pass after the launch before the object returns to ground level? After placing the answer in simple form and reducing, use your calculator to round the answer to the nearest tenth of a second.

Answer

11.3 seconds

34) An object is launched vertically and its height y (in feet) above ground level is given by the equation $y = 192 + 288t - 16t^2$, where t is the time (in seconds) that has passed since its launch. How much time must pass after the launch before the object returns to ground level? After placing the answer in simple form and reducing, use your calculator to round the answer to the nearest tenth of a second.

35) A manufacturer's revenue R accrued from selling x widgets is given by the equation $R = 6000x - 5x^2$. The manufacturer's costs for building x widgets is given by the equation $C = 500000 + 5.25x$. The break even point for the manufacturer is defined as the number of widgets built and sold so the the manufacturer's revenue and costs are identical. Find the number of widgets required to be built and sold so that the manufacturer "breaks even." Round your answers to the nearest widget.

Answer

90 widgets, 1109 widgets

36) A manufacturer's revenue R accrued from selling x widgets is given by the equation $R = 4500x - 15.25x^2$. How many widgets must be sold so that the manufacturer's revenue is $\$125,000$? Round your answers to the nearest widget.

37) Mike gets on his bike at noon and begins to ride due north at a constant rate of 6 miles per hour. At 2:00 pm, Todd gets on his bike at the same starting point and begins to ride due east at a constant rate of 8 miles per hour. At what time of the day will they be 60 miles apart (as the crow flies)? Don't worry about simple form, just report the time of day, correct to the nearest minute.

Answer

7:12 pm

38) Mikaela gets on her bike at noon and begins to ride due north at a constant rate of 4 miles per hour. At 1:00 pm, Rosemarie gets on her bike at the same starting point and begins to ride due east at a constant rate of 6 miles per hour. At what time of the day will they be 20 miles apart (as the crow flies)? Don't worry about simple form, just report the time of day, correct to the nearest minute.

39) The area of a rectangular field is 76 square feet. The length of the field is 7 feet longer than its width. Find the dimensions of the field, correct to the nearest tenth of a foot.

Answer

5.9 by 12.9 feet

40) The area of a rectangular field is 50 square feet. The length of the field is 8 feet longer than its width. Find the dimensions of the field, correct to the nearest tenth of a foot.

41) Mean concentrations of carbon dioxide over Mauna Loa, Hawaii, are gathered by the Earth System Research Laboratory (ESRL) in conjunction with the National Oceanic and Atmosphere Administration (NOAA). Mean annual concentrations in parts per million for the years 1962, 1982, and 2002 are shown in the following table.

Year	1962	1982	2002
Concentration (ppm)	318	341	373

A quadratic model is fitted to this data, yielding

$$C = 0.01125t^2 + 0.925t + 318$$

where t is the number of years since 1962 and C is the mean annual concentration (in parts per million) of carbon dioxide over Mauna Loa. Use the model to find the year when the mean concentration of carbon dioxide was 330 parts per million. Round your answer to the nearest year.

Answer

1973

42) The U.S. Census Bureau provides historical data on the number of Americans over the age of 85.

Year	1970	1990	2010
Population over 85 (millions)	1.4	3.0	5.7

A quadratic model is fitted to this data, yielding

$$P = 0.01375t^2 + 0.0525t + 1.4$$

where t is the number of years since 1970 and P is number of Americans (in millions) over the age of 85. Use the model to find the year when the number of Americans over the age of 85 was $2,200,000$. Round your answer to the nearest year.

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Index

A
ac method
  6.3: Factoring ax² + bx + c when a =1

B
binomial
  5.2: Polynomials

C
Cartesian coordinate plane
  3.1: Graphing Equations by Hand
coefficient
  5.2: Polynomials
completing the square
  8.3: Completing the Square

D
decimal notation
  1.4: Decimal Notation
degree of a polynomial
  5.2: Polynomials
divisor
  6.1: The Greatest Common Factor
domain
  5.1: Functions

E
elimination
  4.3: Solving Systems by Elimination

F
factor
  6.1: The Greatest Common Factor
factoring
  6: Factoring
factoring by grouping
  6.1: The Greatest Common Factor
FOIL
  5.7: Special Products
functions
  5.1: Functions

G
greatest common factor
  6.1: The Greatest Common Factor

I
inequalities
  2.6: Inequalities
interval notation
  2.6: Inequalities

M
monomial
  5.2: Polynomials
multiplying polynomials
  5.6: Multiplying Polynomials

O
ordered pairs
  3.1: Graphing Equations by Hand

P
polynomial
  5: Polynomial Functions
Polynomials
  5.2: Polynomials
Pythagorean theorem
  8.2: Simplifying Radical Expressions

Q
quadratic formula
  8.4: The Quadratic Formula

R
radical expressions
  8.2: Simplifying Radical Expressions
rate
  3.3: Rates and Slope

S
scientific notation
  7.2: Scientific Notation
Simple Radical Form
  8.2: Simplifying Radical Expressions
slope
  3.3: Rates and Slope
standard form of a line
  3.6: Standard Form of a Line
Substitution Method
  4.2: Solving Systems by Substitution
systems of linear equations
  4: Systems of Linear Equations

T
term of a polynomial function
  5.2: Polynomials
trinomial
  5.2: Polynomials

Z
Zero Product Property
  6.2: Solving Nonlinear Equations

Glossary

Sample Word 1 | Sample Definition 1

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    - 2: Solving Linear Equations and Inequalities - *CC BY-NC-ND 3.0*
        - 2.E: Solving Linear Equations and Inequalities (Exercises) - *CC BY-NC-ND 3.0*
        - 2.1: Solving Equations - One Step - *CC BY-NC-ND 3.0*
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        - 2.3: Clearing Fractions and Decimals - *CC BY-NC-ND 3.0*
        - 2.4: Formulae - *CC BY-NC-ND 3.0*
        - 2.5: Applications - *CC BY-NC-ND 3.0*
        - 2.6: Inequalities - *CC BY-NC-ND 3.0*
    - 3: Introduction to Graphing - *CC BY-NC-ND 3.0*
        - 3.E: Introduction to Graphing (Exercises) - *CC BY-NC-ND 3.0*
        - 3.1: Graphing Equations by Hand - *CC BY-NC-ND 3.0*
        - 3.2: The Graphing Calculator - *CC BY-NC-ND 3.0*
        - 3.3: Rates and Slope - *CC BY-NC-ND 3.0*
        - 3.4: Slope-Intercept Form of a Line - *CC BY-NC-ND 3.0*
        - 3.5: Point-Slope Form of a Line - *CC BY-NC-ND 3.0*
        - 3.6: Standard Form of a Line - *CC BY-NC-ND 3.0*
    - 4: Systems of Linear Equations - *CC BY-NC-ND 3.0*
        - 4.E: Systems of Linear Equations (Exercises) - *CC BY-NC-ND 3.0*
        - 4.1: Solving Systems by Graphing - *CC BY-NC-ND 3.0*
        - 4.2: Solving Systems by Substitution - *CC BY-NC-ND 3.0*
        - 4.3: Solving Systems by Elimination - *CC BY-NC-ND 3.0*
        - 4.4: Applications of Linear Systems - *CC BY-NC-ND 3.0*
    - 5: Polynomial Functions - *CC BY-NC-ND 3.0*
        - 5.E: Polynomial Functions (Exercises) - *CC BY-NC-ND 3.0*
        - 5.1: Functions - *CC BY-NC-ND 3.0*
        - 5.2: Polynomials - *CC BY-NC-ND 3.0*
        - 5.3: Applications of Polynomials - *CC BY-NC-ND 3.0*
        - 5.4: Adding and Subtracting Polynomials - *CC BY-NC-ND 3.0*
        - 5.5: Laws of Exponents - *CC BY-NC-ND 3.0*
        - 5.6: Multiplying Polynomials - *CC BY-NC-ND 3.0*
        - 5.7: Special Products - *CC BY-NC-ND 3.0*
    - 6: Factoring - *CC BY-NC-ND 3.0*
        - 6.E: Factoring (Exercises) - *CC BY-NC-ND 3.0*
        - 6.1: The Greatest Common Factor - *CC BY-NC-ND 3.0*
        - 6.2: Solving Nonlinear Equations - *CC BY-NC-ND 3.0*
        - 6.3: Factoring $ax^2 + bx + c$ when $a = 1$ - *CC BY-NC-ND 3.0*
        - 6.4: Factoring $ax^2 + bx + c$ when $a \neq 1$ - *CC BY-NC-ND 3.0*

    - 6.5: Factoring Special Forms - *CC BY-NC-ND 3.0*
    - 6.6: Factoring Strategy - *CC BY-NC-ND 3.0*
    - 6.7: Applications of Factoring - *CC BY-NC-ND 3.0*
- 7: Rational Expressions - *CC BY-NC-ND 3.0*
    - 7.E: Rational Expressions(Exercises) - *CC BY-NC-ND 3.0*
    - 7.1: Negative Exponents - *CC BY-NC-ND 3.0*
    - 7.2: Scientific Notation - *CC BY-NC-ND 3.0*
    - 7.3: Simplifying Rational Expressions - *CC BY-NC-ND 3.0*
    - 7.4: Solving Rational Equations - *CC BY-NC-ND 3.0*
    - 7.5: Direct and Inverse Variation - *CC BY-NC-ND 3.0*
- 8: Quadratic Functions - *CC BY-NC-ND 3.0*
    - 8.E: Quadratic Functions (Exercises) - *CC BY-NC-ND 3.0*
    - 8.1: Introduction to Radical Notation - *CC BY-NC-ND 3.0*
    - 8.2: Simplifying Radical Expressions - *CC BY-NC-ND 3.0*
    - 8.3: Completing the Square - *CC BY-NC-ND 3.0*
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